Fernando, *A converse to Cartan's Theorem B: The extension property for real analytic and Nash sets*

This is joint work with Riccardo Ghiloni.

In 1957 Cartan proved his celebrated Theorem B and deduced that if $\Omega \subset \mathbb{R}^n$ is an open set and X is a coherent real analytic subset of Ω, then X has the analytic extension property, that is, each real analytic function on X extends to a real analytic function on Ω. So far, the converse implication in its full generality remains unproven. As a matter of fact, in the literature only special cases of non-coherent real analytic sets $X \subset \Omega$ without the extension property appear, some of them due to Cartan himself: mainly real analytic sets $X \subset \Omega$ that have a 'tail', that is, X has a non-pure dimensional irreducible analytic component Y such that the set of points of lower dimension of Y is visible inside X (this means that there exists a point $x \in Y$ such that $\dim(X_x) < \dim(Y)$). This kind of examples generated a general feeling that non-coherent real analytic sets without the analytic extension property may have 'tails'.

The aim of this talk is to show that this is not the case and in fact that the converse to Cartan's Theorem B is true in its full generality:

Theorem 1. *If $X \subset \Omega$ has the analytic extension property, then it is a coherent real analytic subset of Ω. In addition, if $X \subset \Omega$ is a non-coherent global analytic set, there exist 'many' meromorphic functions on Ω that are analytic on X, but have no analytic extension to Ω.*

Thus, the class of sets with the analytic extension property coincides with the one of coherent real analytic sets. To prove this fact it is crucial the use of Serre's cohomology of sheaves of analytic function germs and, in particular, to analyze the first cohomology group of the sheaf of zero ideals of X.

The previous characterization can be extended to the Nash case, which is more sophisticated, because of its finiteness properties and its disappointing behavior with respect to cohomology of sheaves of Nash function germs. However, the information obtained in the analytic case allows us to prove the following:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be an open semialgebraic set and let $X \subset \Omega$ be a subset. Each Nash function on X extends to a Nash function on Ω if and only if $X \subset \Omega$ is a coherent Nash set. In addition, if $X \subset \Omega$ is a non-coherent Nash set, there exist 'many' Nash meromorphic functions on Ω that are local Nash on X, but have no Nash extension to Ω*

The 'if' implication goes back to some celebrated results of Coste, Ruiz and Shiota, which can be understood as the Nash counterpart to Cartan's Theorem B in the analytic setting. Again, as it happened in the analytic case, the 'only if' implication had been treated only for Nash sets X that have a 'tail'.

Recall that if $M \subset \mathbb{R}^n$ is a Nash manifold, $\mathcal{C}^\infty$ semialgebraic functions on M coincide with Nash functions on M. As an application of our previous strategies, we confront the coherence of a Nash set $X \subset \Omega$ with the fact that each $\mathcal{C}^\infty$ semialgebraic function on X is a Nash function on X. Namely, we prove that these two properties are equivalent for Nash sets. More generally, we provide a full characterization of the semialgebraic sets $S \subset \Omega$ for which $\mathcal{C}^\infty$ semialgebraic functions on S coincide with Nash functions on S.