Becker, *A Waring Problem in real function fields over* $\mathbb{R}$

The classical Waring Problem deals with the representation of every natural numbers as a sum of a fixed number $g(n)$ of nth powers of natural numbers, for each exponent n. That this is possible was first proven by Hilbert in 1909.

In this talk I deal with an adaptation of the classical Waring Problem to the case of a real function field $F/\mathbb{R}$ of d variables and Pythagoras number p. Let H denote the real holomorphy ring of F, defined as the intersection of all real valuation rings of F, and $\mathbb{E}_+ := H^* \cap \sum F^2$ its group of totally positive units. It can be proven:

1. for each exponent n there exists a bound s such that every element of $\mathbb{E}_+$ can be written as a sum of at most s n-th powers of elements of $\mathbb{E}_+$,
2. let w_n denote the minimal bound for the exponent n then
$$w_2 \leq p, \quad w_n \leq \binom{2n + w_2}{2n}, \quad w_2 = p = 2 \text{ if } d = 1,$$
3. $\mathbb{E}_+$is the largest subset of F^* with the "Waring property" as described in the first statement above.

The talk is focused on the proof of the inequality $w_2 \leq p$, the proofs of the other statements will be sketched. The proof of this inequality makes use of results on a representation $S^k(H) \to C(M, S^k)$ where M denotes the space of real places of F, secondly on the description of H and M by all smooth affine models X of F with a compact real locus $X(\mathbb{R})$ and the study of the corresponding representations $S^k(\mathcal{O}(X)) \to C(X(\mathbb{R}), S^k)$, finally on a recent result of W. Kucharz on rings of continuous rational functions.

References

[1] E. Becker. Sums of powers in real function fields and topology. *Structures algebriques ordonnées, Séminaire 2018–20.*

[2] E. Becker. A note on approximation and homotopy in $C(X, S^n)$, $n = 1, 3, 7$. *Ann. Pol. Math.* 126, 2021.