Orthogonal types to the value group and descent

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What is this talk about?

1. Introduction to stably dominated types and generically stable types.
2. A statement of descent (an improved version!)
3. Introduction to orthogonal types to the value group,
4. In ACVF the following are equivalent:
    - p is generically stable,
    - p is stably dominated,
    - p is orthogonal to the value group.
5. **Leading question:**
   could we generalize the above result in an *AKE*-style for henselian valued fields of equicharacteristic zero?

Notation

- Let $v : K \to \Gamma \cup \{\infty\}$ be a valuation.
  1. $\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$ is the valuation ring,
  2. $\mathcal{M} = \{x \in K \mid v(x) > 0\}$ is its maximal ideal.

  We refer to $\mathrm{k} = \mathcal{O}/\mathcal{M}$ as the *residue field* and we call Γ the *value group*.
- Let M be a henselian valued field of equicharacteristic zero and S be a sort. We call $S(\mathrm{A}) = S \cap \mathrm{dcl}^{eq}(\mathrm{A})$ **the trace of** A **in** S.

Previous work: on ACVF

Motivation: Haskell, Hrushovski and Macpherson introduced the theory of *Stable domination* and set ACVF as a prime example.

Definition (Stable part of the structure)

Let $\mathcal{U}$ be the monster model of a complete theory T with elimination of imaginaries and let A be a set of parameters. We define St_{A} the multi-sorted structure $(D_i, R_j)_{i\in I, j\in J}$ where:

- The sorts D_i are A-definable sets, stable and stably embedded subsets of $\mathcal{U}$;
- For each finite set of the sorts D_i we add all the A-definable relations in their union as $\varnothing$-definable relations R_j.

Remark: St_{A} **is a stable, stably embedded structure.**

A closer look at the stable part of the structure in ACVF

In ACVF there are essentially 3 ways to think about the stable part of the structure over $\mathrm{A} = \mathrm{acl}^{eq}(\mathrm{A})$ a set of imaginary parameters.

1. St_{A} as defined before.
2. $\mathrm{Int}(k, \mathrm{A})$ the collection of the A-definable sets that are internal to the residue field k.
3. As a linear structure $Lin_{\mathrm{A}} = \bigcup_{s \in \mathrm{dcl}^{eq}(\mathrm{A})} s/\mathcal{M}s$, where $s \subseteq K^n$ is a $\mathcal{O}$-lattice.
   Indeed, $s/\mathcal{M}s$ is a finite dimensional vector space over the residue field.

Stably dominated types

Definition

Let p be a global A-invariant type. We say that p is *Stably dominated over* $A \subseteq B$ if there is an B-(pro)-definable map $f : p \to St_B$ that dominates p over B.

Stably dominated types

Definition

Let p be a global A-invariant type. We say that p is *Stably dominated over* $\mathrm{A} \subseteq \mathrm{B}$ if there is an B-(pro)-definable map $\mathrm{f} : \mathrm{p} \to \mathrm{St}_{\mathrm{B}}$ that dominates p over B.

This is if $a \models \mathrm{p} \restriction_{\mathrm{B}}$ and $\mathrm{f}(a) \downarrow_{\mathrm{B}} d$ then $a \downarrow_{\mathrm{B}} d$.

e.g. In ACVF*: let* p *the generic of the valuation ring* $\mathcal{O}$ *it is dominated by res.*

This is highly sensitive to the set of parameters: base B and f is B-definable so it takes values on St_{B}!

Change of base

Theorem (HHM)

(Change of base) Let $\mathrm{A} \subseteq \mathrm{B}$ *and* p *be a global* A*-invariant type. Assume that:*

Invariant Extension property (IEP): $\mathrm{tp}(\mathrm{B}/\mathrm{A})$ **has a global** *Aut*$(\mathcal{U}/\mathrm{A})$-**invariant extension** q.

Then:

- *Descent:* p *is stably dominated over* B*, then* p *is stably dominated over* A*. (very technical!)*
- *Going up:* p *is stably dominated over* A *then it is stably dominated over* B*. (easy direction)*

Descent: an obstruction

- Haskell, Hrushovski, Macpherson[2006]: *"The hypothesis IEP for descent is stronger than it might be; it would be good to investigate the weakest assumptions under which some version of the Theorem could be proved".*

- Hrushovski,Rideau-Kikuchi [2018]: *Can descent be proved without the additional hypothesis that* $\mathrm{tp}(\mathrm{B}/\mathrm{A})$ *has a global* A*-invariant extension?*

Answer (joint with P. Simon): Yes! it can be done, and actually for valued fields the argument is much simpler.

Descent: The general version

Theorem (Simon, V.)

(T be any theory)
Let p be a global A-invariant type and let b be such that p is stably dominated over $\mathrm{A}b$. Then p is stably dominated over A.

generically stable types

Intuition: Those types that behave like types in stable theories.

Remark:
If p *is stably dominated* $\rightarrow$ p is generically stable.

Definition (T any theory)

A global type $\mathrm{p}(x)$ is generically stable over A if it is A-invariant and for every ordinal $\alpha \geq \omega$, any Morley sequence $(a_i)_{i<\alpha}$ of p over A (i.e $a_i \models \mathrm{p} \restriction_{\mathrm{A}a_{<i}}$) and any $\mathcal{L}(M)$ formula $\phi(x)$, the set $\{i < \alpha \mid M \models \phi(a_i)\}$ is finite or co-finite.

generically stable types

Intuition: Those types that behave like types in stable theories.

Some few remarks:
Let p be a generically stable type over A and $a \models \mathrm{p}\restriction_{\mathrm{A}}$.

1. $\mathrm{p}\restriction_{\mathrm{A}}$ is stationary and p is its only global non-forking extension;
2. p is definable over A;
3. (Symmetry) If $b \downarrow_{\mathrm{A}} \mathrm{A}$, then $a \downarrow_{\mathrm{A}} b$ if and only if $b \downarrow_{\mathrm{A}} a$;
4. (Right transitivity) $a \downarrow_{\mathrm{A}} b$ and $a \downarrow_{\mathrm{A}b} d$ if and only if $a \downarrow_{\mathrm{A}} bd$.

Orthogonal types to the value group

Definition (Orthogonal types)

Let p be a global A-invariant type and Γ be a stably embedded sort. We say that p is *orthogonal* to Γ if:

- for any set of parameters $\mathrm{A} \subseteq \mathrm{B}$ and $a \models \mathrm{p}_{|\mathrm{B}}$, $\Gamma^{eq}(\mathrm{B}a) = \Gamma^{eq}(\mathrm{B})$.
- (equivalently) any definable map $\mathrm{f} : \mathrm{p} \to \Gamma^{eq}$, then $\mathrm{f}(\mathrm{p})$ is constant.

Characterization of orthogonal types in ACVF

Proposition (HHM)

Given a global A*-invariant type* p *in* ACVF*, the following are equivalent:*

- p *is generically stable,*
- p *is stably dominated,*
- p *is orthogonal to the value group.*

Leading question: Is there a similar characterization of orthogonal types in further henselian valued fields?

The Ax-Kochen/Ershov Theorem

SPINE PHILOSOPHY: AX-KOCHEN/ERSHOV PRINCIPLE

Theorem (Ax-Kochen/Ershov)

Let (K, k, Γ) and (K', k', Γ') be two henselian valued fields of equicharacteristic zero, then $K \equiv K'$ if and only if $k \equiv_{fields} k'$ and $\Gamma \equiv_{OAG} \Gamma'$.

Principle

The model theory of a henselian valued field of equicharacteristic zero is controlled by its residue field and its value group.

Towards a characterization of orthogonal types in valued fields

Proposition (Intuitive Goal)

Let (K, v) *be a henselian valued field and* p *be a global* A*-invariant type centered in the main field. The following are equivalent:*

1. p *is orthogonal to the value group,*
2. p *is fully controlled by the residue field,*
3. *Its reduct to* ACVF *is stably dominated (equivalently, generically stable and orthogonal to the value group).*

What does really correspond to the "residual part" as an analogue of the stable part?

We introduced what we called the structure $\mathrm{Lin}_{\mathrm{A}}$.

Definition

1. Let Mod be the collection of codes $\mathcal{O}$-modules of K^n.
2. For each $s \in \mathrm{Mod}$, the reduction $\mathrm{red}(s) = s/\mathcal{M}s$ is a finite dimensional vector space over k.
3. Let $\mathrm{Lin} = \bigcup_{s \in \mathrm{Mod}} s/\mathcal{M}s$.
4. Let $\mathrm{A} = \mathrm{acl}^{eq}(\mathrm{A}) \subseteq M^{eq}$,

$$\mathrm{Lin}_{\mathrm{A}} = \bigcup_{s \in \mathrm{Mod}(\mathrm{A})} s/\mathcal{M}s \text{ with the A-induced structure.}$$

The generalized geometric sorts: $\mathcal{G} = \mathrm{K} \cup \mathrm{k} \cup \Gamma^{eq} \cup \mathrm{Mod} \cup \mathrm{Lin}$.

A general definition of residual domination

Warning! we will work with 3 different structures

Setting:

- Let M be an $\mathrm{k} \cup \Gamma$-expansion of a model of $Hen_{0,0}$ in some language $\mathcal{L}$ (e.g. 3-sorted language (K, RV, Γ) with Γ Morleyized). Assume dense value group.
- Let M_{ur} be its maximal unramified extension.
- Let M^{alg} be its field algebraic closure.

Note:

- $\mathcal{G} = \mathrm{K} \cup \mathrm{Mod} \cup \mathrm{Lin}$ and these are sorts that both M and M_{ur} can see.
- Let Geo be the usual geometric sorts in ACVF, then $\mathrm{Geo} \subseteq \mathcal{G}$.
- let $M \prec N$ and $a \in K(N)$.
  $\mathrm{tp}_{\mathrm{ACVF}}(a/M)$ denotes the quantifier free type of a in N^{alg} over M (seen as a set of parameters).

A definition for residual domination: the general case

Recall: $\mathcal{G} = \mathrm{K} \cup \mathrm{Mod} \cup \mathrm{Lin}$ and these are sorts that both M and M_{ur} can see. And $\mathrm{Geo} \subseteq \mathcal{G}$.

Definition (Residual domination)

Let $\mathrm{A} = \mathcal{G}(\mathrm{acl}^{eq}(\mathrm{A})) \subseteq M^{eq}$. Let $M \prec N$ and $a \in \mathrm{K}(N)$.
Let τ_{A} be the $\mathcal{L}(\mathrm{A})$-pro definable map that sends a to an enumeration of $\mathrm{Lin}_{\mathrm{A}}(\mathrm{A}a) = \mathrm{Lin}_{\mathrm{A}} \cap \mathrm{dcl}^{eq}(\mathrm{A}a)$.
We say that $\mathrm{tp}(a/\mathrm{A})$ is *residually dominated via* τ_{A} if for every $\mathrm{B} = \mathrm{acl}^{eq}(\mathrm{AB}) \subseteq M^{eq}$

$$\tau_{\mathrm{A}}(a) \mathop{\downarrow}_{\mathrm{A}}^{ur} \mathrm{Lin}_{\mathrm{A}}(\mathrm{B}), \text{ implies } \mathrm{tp}(a/\mathrm{A}) \cup \mathrm{tp}(\tau_{\mathrm{A}}(a)/\mathrm{B}) \vdash \mathrm{tp}(a/\mathrm{B})$$ [1]

Where $\downarrow^{ur}$ denotes non forking independence in the stable structure $\mathrm{Lin}_{\mathrm{A}}(M_{ur})$.

[1] (equivalently $\mathrm{tp}(\mathrm{B}/\mathrm{A}\tau_{\mathrm{A}}(a)) \vdash \mathrm{tp}(\mathrm{B}/\mathrm{A}a)$)

Characterizing orthogonal types to the value group

Proposition (Cubides,Rideau-Kikuchi,V.)

Let $\mathrm{A} = \mathrm{acl}^{eq}(\mathrm{A}) \subseteq M^{eq}$, $M \prec N$, $a \in \mathrm{K}(N)$ *and* $\mathrm{p} = \mathrm{tp}(a/M)$.
Assume that:

- $\mathrm{tp}_{\mathrm{ACVF}}(a/M)$ *is* $\mathcal{L}(\mathrm{A})$*-definable (in* M*).*

Then the following are equivalent:

- $\mathrm{tp}(a/\mathrm{A})$ *is residually dominated (in* M*),*
- p *is orthogonal to the value group (in* M*),*
- $\mathrm{tp}_{\mathrm{ACVF}}(a/M^{alg})$ *generically stable and its canonical base is contained in* $\mathrm{Geo}(\mathrm{A})$*. (equivalently, it is stably dominated, orthogonal to* Γ *in* ACVF*.)*

Change of base

Proposition (Rideau-Kikuchi, Cubides, V.)

(Change of base) Let $\mathrm{A} = \mathrm{acl}^{eq}(\mathrm{A}) \subseteq M$, $M \prec N$ *and* $a \in K(N)$.

- *Assume that* $\mathrm{tp}_{\mathrm{ACVF}}(a/M)$ *is* $\mathcal{L}(\mathrm{A})$*-definable (in* M*).*

Let $\mathrm{Geo}(\mathrm{A}) \subseteq \mathrm{B} = \mathrm{acl}^{eq}(\mathrm{B}) \subseteq \mathrm{D} = \mathrm{acl}^{eq}(\mathrm{D}) \subseteq M^{eq}$.

1. *(Descent) If* $\mathrm{tp}(a/\mathrm{D})$ *is residually dominated. Then* $\mathrm{tp}(a/\mathrm{B})$ *is residually dominated.*
2. *(Going up) If* $\mathrm{tp}(a/\mathrm{B})$ *is residually dominated, then* $\mathrm{tp}(a/\mathrm{D})$ *is residually dominated.*

Valued fields with operators

Consider an expansion $\widetilde{M}$ in a language $\widetilde{L}$ of an equicharacteristic zero henselian valued field M with dense valued group. Let $\mathrm{f}:\mathrm{K}\to\mathrm{K}$ be a pro- definable function such that:

(**H**) for every finite tuple $a\in\mathrm{K}(\widetilde{M})$, $\mathrm{tp}_{\mathcal{L}}(\mathrm{f}(a))\vdash\mathrm{tp}_{\widetilde{\mathcal{L}}}(a)$.

This includes several cases of valued fields with operators:

- valued difference fields with $\mathrm{f}(x)=(\sigma^i(x))_{i\geq 0}$ (Durhan-Onay) (includes VFA_0)
- δ-henselian valued fields with a monotone derivation (Scanlon),
- $\mathrm{Hen}_{(0,0)}$ with generic derivations (Fornasiero, Terzo).

A characterization of the orthogonal types

Theorem (Rideau-Kikuchi, Cubides, V.)

Let $\widetilde{M}$ satisfying hypothesis (H) and $\widetilde{M} \prec \widetilde{N}$ and $a \in K(\widetilde{N})$.
Assume:

- *the residue field and Γ are stably embedded and orthogonal to each other.*
- $\mathrm{p} = \widetilde{\mathrm{tp}}(a/\widetilde{M})$ *is* A*-invariant,*
- $\mathrm{tp}_{\mathrm{ACVF}}(\mathrm{f}(a)/M)$ *is* $\mathcal{L}(\mathrm{A})$*-definable,*

Then, the following are equivalent:

- p *is residually dominated for any* $\mathrm{Geo}(\mathrm{A}) \subseteq \mathrm{D} = \widetilde{\mathrm{acl}(\mathrm{D})}$.
- p *is orthogonal to* Γ,
- $\mathrm{tp}_{\mathrm{ACVF}}(\mathrm{f}(a)/M^{alg})$ *is generically stable in* ACVF *and its canonical base is contained in* $\mathrm{Geo}(\mathrm{A})$ *(equivalently stably dominated, residually dominated, orthogonal to* Γ*),*

Lifting model theoretic tameness of the residue field

Corollary

Let $\widetilde{M}$ satisfying H.
Let $a \in \mathrm{K}^n$ and assume:

- $\mathsf{tp}_{\mathrm{ACVF}}(\mathrm{f}(a)/M)$ *is* $\mathcal{L}(\mathrm{A})$*-definable,*
- $\mathrm{p} = \mathsf{tp}(a/M)$ *is* A*-invariant,*
- *the residue field is stable.*

Then the following are equivalent:

- p *is orthogonal to* Γ *iff* p *is residually dominated iff its reduct to* ACVF *is generically stable.*
- p *is generically stable.*

Generically simple types

Definition

Let T be NTP_2 and let A be an arbitrary set of parameters and $\mathrm{p} \in \mathcal{S}(\mathrm{A})$. Then p is generically simple if for every $b \in \mathcal{U}$ and $a \models \mathrm{p}$ if $b \downarrow_{\mathrm{A}} a$ implies $a \downarrow_{\mathrm{A}} b$.

Proposition (Chernikov-Simon)

The following statements hold:

1. *If* $\mathrm{tp}(a/\mathrm{A}b)$ *and* $\mathrm{tp}(b/\mathrm{A})$ *are generically simple then* $\mathrm{tp}(ab/\mathrm{A})$ *is generically simple.*
2. *If* $\mathrm{tp}(a_i/\mathrm{A})$ *is generically simple and* $a_i \downarrow_{\mathrm{A}} a_{<i}$*, then* $a_i \downarrow_{\mathrm{A}} a_{\neq i}$*.*
3. *If* A *is an extension base:*
    1. *Let* $\mathrm{A} \subseteq \mathrm{B}$ *and* $\mathrm{q} \in \mathcal{S}^n(\mathrm{B})$ *if* $\mathrm{q}_{|\mathrm{A}}$ *is generically simple so is* q*.*
    2. *If* $\mathrm{tp}(a/\mathrm{A})$ *is generically simple and* $a \downarrow_{\mathrm{A}} b$ *then* $b \downarrow_{\mathrm{A}} a$*.*

Orthogonal types: inherit the model theoretic tameness of the residue field

Corollary

Let $\widetilde{M}$ satisfying H.
Let $a \in \mathrm{K}^n$ and assume:

- $\mathrm{tp}_{\mathrm{ACVF}}(\mathrm{f}(a)/M)$ *is* $\mathcal{L}(\mathrm{A})$*-definable,*
- $\mathrm{p} = \mathrm{tp}(a/M)$ *is* A*-invariant,*
- $\mathrm{Th}(\widetilde{M})$ *is* NTP_2*, the residue field is simple and every set of parameters* $\mathrm{acl}^{eq}(\mathrm{A})$ *is an extension basis.*

Then the following are equivalent:

1. p *is orthogonal to* Γ *iff* p *is residually dominated iff its reduct to* ACVF *is stably dominated.*
2. p *is generically simple.*

Thanks a lot!

For:

- the invitation,
- and your attention :-).