Update on decidability of real closed rings

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1. This talk is about...

...rings of continuous definable functions, like

$$C(X) = \{f : X \longrightarrow R \mid f \text{ continuous and semi-algebraic}\},$$

where R is a real closed field and $X \subseteq R^n$ is semi-algebraic.

$C(X)$ is a ring under point wise addition and multiplication.
$C(X)$ is partially ordered by $f \leq g :\Longleftrightarrow \forall x \in X : f(x) \leq g(x)$ and this partial order is definable by the formula $\exists z : z^2 = g - f$.

I report on a few updates over the last years with a focus on the case $X = R$ and $R = \mathbb{R}_{\text{alg}} = \mathbb{R} \cap \overline{\mathbb{Q}}$.

Main open question:
Is $C(\mathbb{R}_{\text{alg}})$ decidable in the first-order language of rings?

$\curvearrowright$

2. Undecidability in dimension > 1

(a) The ring of *all* continuous functions $M \longrightarrow \mathbb{R}$ for any non-discrete metric space M is undecidable [Che80].

(b) If $\dim(X) \geq 2$, then $C(X)$ is undecidable, because the zero set lattice

$$L_X = \{\{f = 0\} \mid f \in C(X)\}$$

is interpretable in it (use $\{f = 0\} \subseteq \{g = 0\} \iff g \subseteq \mathrm{JacobsonRad}(f)$), and L_X interprets $(\mathbb{N}, +, \cdot)$ [Grz51; Tre17].

(c) The ring $\{f : \mathbb{Q}_p \longrightarrow \mathbb{Q}_p \mid f$ is continuous and definable in $\mathbb{Q}_p\}$ interprets $(\mathbb{N}, +, \cdot)$ and is thus undecidable, [DT20].
[although the zero set lattice of $\mathbb{Q}_p$ and of $\mathbb{Q}_p^n$ is decidable, [Dar19]]

Other variants consider different codomains (e.g. $\mathbb{Z}$) and abstractions of the rings above, which look much more friendly but are still undecidable.

In order to talk about the matter in a more systematic way I will now talk about the category of *real closed rings* (in the sense of Niels Schwartz, [Sch89]) which is a suitable ambient algebraic context to study these kind of rings:

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3. Real Closed Rings, an elementary definition

Fact Let $f \in C(\mathbb{R}^n_{\mathrm{alg}})$. Then there is some polynomial

$$P_f \in \mathbb{Z}[\bar{X}, Y, \bar{U}],\ \bar{X} = (X_1, \ldots, X_n)$$

such that

(a) The graph of f is defined in $\mathbb{R}_{\mathrm{alg}}$ by $\exists \bar{u}\, P_f(\bar{x}, y, \bar{u}) = 0$, and

(b) for some continuous, semi-algebraic function $s : \mathbb{R}^n_{\mathrm{alg}} \longrightarrow \mathbb{R}^{\bar{u}}_{\mathrm{alg}}$ we have $P_f(\bar{x}, f(\bar{x}), \bar{s}(\bar{x})) = 0$ $(\bar{x} \in \mathbb{R}^n_{\mathrm{alg}})$.

This follows quickly from the finiteness theorem applied to the graph of f.

Definition A ring (commutative and unital) A is **real closed** if it is reduced and for every f and any choice of P_f as in the fact above we have

$$A \models \forall \bar{x}\, \exists y\, \exists \bar{u} : P_f(\bar{x}, y, \bar{u}) = 0.$$

Theorem [Tre07] The ring A is real closed if and only if it is real closed in the sense of Niels Schwartz [Sch89].

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4. Real Closed Rings: Examples

How does this definition work for a real closed ring A?

If $f \in C(\mathbb{R}^n_{\mathrm{alg}})$, then the axioms can be used to define a function $f_A : A^n \longrightarrow A$ and the axioms also guarantee that the f_A's compose exactly like the f's.

Think of f_A as 'scalar extension' of f to A, or as "composition with f".

For example, $\mathbb{R}_{\mathrm{alg}} \subseteq A$ (if $A \neq 0$): For $a \in \mathbb{R}_{\mathrm{alg}}$ the constant function f of value a is in $C(\mathbb{R}_{\mathrm{alg}})$ and satisfies $\mu(f(x)) = 0$ for all x, where μ is the minimal polynomial of a. Then $\mu_A(f_A(x)) = 0$ for all $x \in A$ as well. But $\mu_A = \mu$, as one checks without difficulty using $A \neq 0$, and then $f_A(1) = a$ follows (easily).

Using these functions one then also sees that A is partially ordered by

$$a \leq b \iff \exists c \in A : b - a = c^2.$$

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4. Real Closed Rings: Examples

Examples

(a) A real closed field R is a field that is also a real closed ring.

Here $f_R : R^n \longrightarrow R$ is the function defined by *any* formula defining the graph of $f : \mathbb{R}_{\mathrm{alg}}^n \longrightarrow \mathbb{R}_{\mathrm{alg}}$.

(b) A convex subring V of a real closed field R is a valuation ring that is also a real closed ring.

Here $f_V = f_R|_{V^n}$.

(c) Rings of *all* real valued continuous functions on some topological space as well as the rings $C(X)$ considered above are real closed.

Here $f_{C(X)}(a_1, \ldots, a_n) = f_R \circ (a_1, \ldots, a_n)$.

↷

5. Real Closed Rings: Properties

Properties (I wrote a Wikipedia page)

(a) The category RCR of real closed rings and ringhomomorphism is reflective in the category of commutative rings. In plain English this means that every ring A has a **real closure**, namely a real closed ring $\rho(A)$ and a ringhomomorphism $\rho_A : A \longrightarrow \rho(A)$ such that for every ringhomomorphism $A \longrightarrow B \in \mathtt{RCR}$ there is a unique ring homomorphism $\rho(A) \longrightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccc} \rho(A) & \xrightarrow{\exists!} & B \\ {\scriptstyle \rho_A}\uparrow & \nearrow_{\forall} & \\ A & & \end{array}.$$

Examples: $\rho(R[\bar{x}]) = C(R^n)$, $\rho(\mathbb{C}) = 0$, $\rho(\mathbb{Q}) = \mathbb{R}_{\text{alg}}$, $\rho(\mathbb{Q}(\sqrt{2})) = \mathbb{R}_{\text{alg}} \times \mathbb{R}_{\text{alg}}$.

(b) The category RCR is complete and cocomplete and forms a variety in the sense of universal algebra.

(c) All localizations of real closed rings are real closed and local real closed rings are local Henselian rings (but not nec. domains).

Concretely here: the localization $C(X)_{\mathfrak{m}}$ at a maximal ideal $\mathfrak{m}$ is real closed. If $X = R$ and $\mathfrak{m} = \{f \mid f(0) = 0\}$, this ring is isomorphic to $V \times_R V$, where V is the convex hull of R in $R(t)^{\text{rc}}$, $t > R$.

6. Real Closed Rings: Properties

(d) If R is a real closed field, then

$$\begin{array}{lcl} \mathrm{Sper}(C(R^n)) & \cong & \mathrm{Spec}(C(R^n)), \text{ even as ringed spaces.} \\ S_n(R) & \cong & \mathrm{Spec}(C(R^n))_{\mathrm{patch}} \text{ as topological spaces.} \end{array}$$

Less ad hoc: The category of closed semi-algebraic subsets X of R^n, $n \in \mathbb{N}$ with continuous semi-algebraic functions is anti-equivalent to the full subcategory of rings, consisting of those RCRs that are finitely generated over R as RCRs.

(Not unlike the anti-equivalence of Zariski closed sets defined over an alg. closed field K and the finitely generated K-algebras.)

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7. Root causes for undecidability

If we look at a picture of a real closed ring as ring of global sections over $\mathrm{Spec}(A)$, there are two principal reasons for undecidability of A:

Stalks are undecidable (and interpretable). For example, every archimedean pair of real closed fields is interpretable in a real closed domain of Krull dimension 1, whose spectrum thus consist of exactly two points $\{0\} \subsetneq \mathfrak{m}$. By [Bau82a; Mac68] there are uncountably many complete theories of such pairs. Hence most of them are undecidable and hence there is no hope to get a good grasp on complete theories of such real closed domains either. For a given pair $K \subseteq L$ of real closed fields take $A = K + \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $L(t)^{\mathrm{rc}}$, $t > R$. Such rings are studied under the name *pseudovaluation domains*.

The representation space is undecidable (and interpretable): The lattice of compact open subsets of $\mathrm{Spec}(A)$ is often interpretable in A and these lattices very often interpret $(\mathbb{N}, +, \cdot)$ [Tre17].

Several classes of real closed rings without these obstructions have been shown to have good model theoretic behavior [Mac68; Ast08; Gui25]. Specifically when the zero set lattice is close to be Boolean, the Feferman-Vaught technique on generalized products are available.

8. Semi-algebraic curves

An element a of $C(R)^n$ is a continuous semi-algebraic curve $a : R \longrightarrow R^n$, hence the decidability of $C(R)$ asks about whether the first order theory of semi-algebraic curves with addition and multiplication is decidable.

Here is a seemingly strong argument against it:

Let P be the set of all compact, connected, semi-algebraic subsets of $\mathbb{R}^2$ of dimension ≤ 1. Consider the **poset** (=partially ordered set) $(P, \subseteq)$. This first-order structure interprets $(\mathbb{N}, +, \cdot)$ [Tre17].

[Intermission: This is where the question to A. Berarducci – answered fully in [BG25] – came from: We know, using [Ast13] (among other things), that $(P, \subseteq)$ does *not* interpret the field $\mathbb{R}$ "on a line" and the question was if this might change for algebraically closed fields, where P is replaced by irreducible projective curves. Another driving force here is János Kollár's question on *What determines a variety?* [KLOS23]]

Now, P is precisely the set of all images of semi-algebraic curves $a \in C([0,1]_{\mathbb{R}})^2$.

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However, why would the ring $C([0,1]_{\mathbb{R}})$ interpret this poset?

In fact, the corresponding semilinear question considers the subposet P_{lin} of piecewise linear members of P.

Again, the poset P_{lin} interprets $(\mathbb{N}, +, \cdot)$, but the corresponding 'parametrizing partially ordered group' of all continuous semi-linear functions $[0,1] \longrightarrow \mathbb{R}$ is decidable and has a "tame" set of definable sets.

This will be explained better below.

$\curvearrowright$

9. Decidability of the ordered module on curves

Theorem (Deacon Linkhorn [Lin21], Ricardo Palomino [Pal25])

Let $C = C(\mathbb{R}_{\mathrm{alg}})$ and let M be the C-module C, expanded by the partial order $\leq$. As a first order structure,

$$M = (C, +, \leq, 0, (\alpha \cdot \_ \mid \alpha \in C))$$

in the language of partially ordered C-modules that has a relation symbol $\leq$, a binary function symbol $+$ and unary function symbols, one for each scalar α from C.

Then M is decidable and model complete in a natural language to be explained next.

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10. Decidability of the ordered module on curves: proof

Here $C = C(R)$, R any real closed field. In the sheaf picture:

- the stalks are all either real closed fields or of the form $V \times_R V$ as in property (c) and these are well understood and decidable.
- The zero set lattice of C is the lattice $L(R)$ of closed semi-linear subsets of R, i.e. finite unions of closed intervals of R. This lattice $L(R)$ is already interpretable in the reduct $(C(R), +, \leq)$. (Exercise)

Facts

(1) The lattice $L(R)$ is bi-interpretable with weak monadic second order logic of the chain $(R, \leq)$. In the terminology of A. Berarducci's talk this is the 2-sorted structure $((R, \leq), \mathrm{Fin(R)}, \in)$. From automaton theory we then know that this structure is decidable [Läu68].

(2) Linkhorn has then used ideas from Shelah's composition method to obtain a model completeness result for $L(R)$, which reads as follows: $L(R)$ is model complete in the language with a constant symbol for $\emptyset$ and primitive functions

$$x \cap y, x \cup y, \min(x), \max(x), \mathrm{LeftEndpoints}(x), \mathrm{RightEndpoints}(x).$$

↷

10. Decidability of the ordered module on curves: proof

R. Palomino has reduced the model theory of the partially ordered module $M = (C, +, \leq, 0, (\alpha \cdot \_ \mid \alpha \in C))$ to the zero set lattice $L(R)$ and the valuation ring $V =$ convex hull of R in $R(t)^{\mathrm{rc}}$, $t > R$.
This goes as follows. For each minimal prime ideal $\mathfrak{q}$ of $C(R)$ that is not maximal, let $\pi_{\mathfrak{q}} : C(R) \longrightarrow V = C(R)/\mathfrak{q}$ be the residue map. Consider the many sorted structure

$$M_{\mathrm{val}} := (M, (L(R), \subseteq), (V, +, \cdot))$$

with the following maps between sorts: $P : M \longrightarrow L(R), P(f) = \{f \geq 0\}$, and for each $\mathfrak{q}$ as a above, the map $\pi_{\mathfrak{q}} : C \longrightarrow V$.

Then M_{val} has quantifier elimination in the home sort M. The procedure is recursive and explicit.

The sorts $L(R)$ and V are stably embedded. Hence by Cherlin-Dickmann (dealing with the V-sort) and Linkhorn's result dealing with the L-sort we get existential definitions of any M-definable set in the language of M_{val}.

(The proof substantially generalizes unpublished results announced by Shen and Weispfenning in [SW87] on divisible abelian ℓ-groups).

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