Degree Bounds in Hilbert's 17th Problem

DDG40

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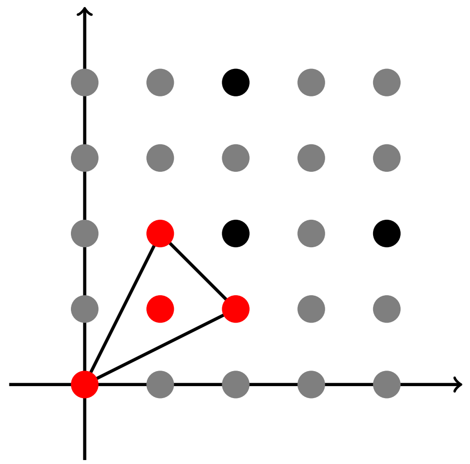
Hilbert's 1888 Theorem

Theorem (Hilbert, 1888). *Given $n, d \in \mathbb{N}$, every nonnegative homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d is a sum of squares of polynomials if and only if (1) $n = 2$, (2) $d = 2$, or (3) $(n, d) = (3, 4)$.*

Example. *The **Motzkin polynomial** $x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \in \mathbb{R}[x, y, z]$ is nonnegative but not a sum of squares.*

$$\frac{x^4y^2 + x^2y^4 + z^6}{3} \geq \sqrt[3]{x^4y^2 \cdot x^2y^4 \cdot z^6} = x^2y^2z^2$$



Hilbert's 1893 Theorem

Theorem (Hilbert, 1893). *For every nonnegative form* $f \in \mathbb{R}[x, y, z]$ *of degree* $2d$ *there is a nonnegative polynomial* g *of degree* $2d - 4$ *such that* $g f$ *is a sum of squares of polynomials.*

Example. $2d = 10$*: If* f *is a nonnegative homogeneous polynomial in* $\mathbb{R}[x, y, z]_{10}$ *of degree* 10*, then there is a nonnegative multiplier* $g_1 \in \mathbb{R}[x, y, z]$ *such that* $g_1 f$ *is a sum of* 3 *squares. Recursively, there is a nonnegative multiplier* g_2 *such that* $g_2 g_1$ *is a sum of* 3 *squares so that*

$$(g_2 g_1) f = g_2(g_1 f) = \left(\sum_{i=1}^{3} \ell_i^2\right)\left(\sum_{j=1}^{3} f_j^2\right) = \sum_{i=1}^{4} h_i^2$$

because

$$\begin{aligned}
&(a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2) \\
&= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2)^2 + (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2)^2 \\
&+ (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2)^2 + (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)^2
\end{aligned}$$

Hilbert's 17th Problem

Theorem (Hilbert, 1893). *Every nonnegative polynomial* $f \in \mathbb{R}[x, y]$ *is a sum of 4 squares in* $\mathbb{R}(x, y)$.

Hilbert's 17th Problem: Is every $f \in \mathbb{R}[x_1, \ldots, x_n]$ with $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ a sum of squares of rational functions?

$$f = \left(\frac{p_1}{q_1}\right)^2 + \left(\frac{p_2}{q_2}\right)^2 + \ldots + \left(\frac{p_r}{q_r}\right)^2$$

Equivalently, does there exist a polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ such that $h^2 f$ is a sum of squares $p_1^2 + p_2^2 + \ldots + p_r^2$ of polynomials.

Theorem (Artin, 1927). *Yes, every nonnegative polynomial is a sum of squares of rational functions.*

Hilbert 17: Complexity

Theorem (Hilbert, 1893). *Every nonnegative polynomial* $f \in \mathbb{R}[x, y]$ *is a sum of 4 squares in* $\mathbb{R}(x, y)$.

Theorem (Pfister, 1967). *Every nonnegative rational function* $f \in \mathbb{R}(x_1, \ldots, x_n)$ *is a sum of at most* 2^n *squares.*

Hilbert 17: Degree Bounds

Main Question: What about degree bounds (in dimension 2)?

Theorem (Blekherman, Smith, Velasco, 2019). *There exist nonnegative polynomials $f \in \mathbb{R}[x, y]$ of degree $2j$ such that gf is not a sum of squares for any nonnegative g of degree $2k$ with $k < j - 3$.*

Theorem (Lombardi, Perrucci, Roy, 2020). *An upper bound on the degree of a multiplier g as above from quantifier elimination is a tower of exponentials.*

Example. $2d = 10$: ***Hilbert****'s result: for every nonnegative form $f \in \mathbb{R}[x, y, z]$,* $\deg(f) = 10$*, there is a sum of squares g of degree 8 such that gf is a sum of squares.*
Blekherman, S, Smith, Velasco*: there is a sum of squares h of degree 6 (with special support) such that gf is a sum of squares.*

Gram map

Let $P \subset \mathbb{R}^n$ be a lattice polytope (meaning $P = \operatorname{conv}(P \cap \mathbb{Z}^n)$).
List the monomials x^α for $\alpha \in P \cap \mathbb{Z}^n$ in a vector $m_P \in \mathbb{R}[x_1, \ldots, x_n]^N$. Define the **Gram map**

$$G_P : \begin{cases} \mathbb{R}^{N\times N}_{sym} \to \mathbb{R}[x_1, \ldots, x_n] \\ A \mapsto m_P^\top A \, m_P \end{cases}$$

Example. *Let* $P = \operatorname{conv}(\{(0,0),(0,1),(1,0),(1,1)\}) \subset \mathbb{R}^2$ *so that*

$$G_P : \begin{cases} \mathbb{R}^{4\times 4}_{sym} \to \mathbb{R}[s, x] \\ A \mapsto (1, s, x, sx)\, A \,(1, s, x, sx)^\top \end{cases}$$

Theorem. *A polynomial* $f \in \mathbb{R}[x_1, \ldots, x_n]$ *whose Newton polytope is contained in* $2P$ *is a sum of squares of polynomials if and only if there is a positive semidefinite matrix* A *with* $G_P(A) = f$.

Sums of Squares and Projective Varieties

$$G_P : \begin{cases} \mathbb{R}^{N\times N}_{sym} \to \mathbb{R}[x_1,\dots,x_n] \\ \quad A \mapsto m_P^\top A\, m_P \end{cases}$$

The kernel of G_P is the set of *quadratic relations* among the monomials in P; in terms of toric geometry, $\ker(G_P) = I(X_P)_2$.

Example. *Let* $P = \operatorname{conv}\{(0,0),(2,0),(0,2)\} \subset \mathbb{R}^2$. *Then*

$$X_P = \nu_2(\mathbb{P}^2) = \{(x^2 : xy : xz : y^2 : yz : z^2) \in \mathbb{P}^5 \mid (x : y : z) \in \mathbb{P}^2\}$$

and $I(X_P)_2 = \langle x_0x_3 - x_1^2, x_0x_5 - x_2^2, x_3x_5 - x_4^2, x_1x_2 - x_0x_4, x_1x_4 - x_2x_3, x_2x_4 - x_1x_5\rangle$. *A quadratic form on* $\mathbb{P}^5$ *restricted to* X_P *corresponds to a* ***ternary quartic***.

Multiplier Bounds

Theorem (Hilbert, 1893). *For every nonnegative form $f \in \mathbb{R}[x, y, z]$ of degree d there is a nonnegative polynomial g of degree $d - 4$ such that gf is a sum of squares of polynomials.*

Theorem (Blekherman, Sinn, Smith, Velasco; *). *Let $P, Q \subseteq \mathbb{R}^2$ be lattice polygons such that no integer translate of P is contained in Q. Let h be the total number of reduced connected components of the set differences $P \setminus Q'$ as Q' ranges over all lattice translates of Q. If the inequality*

$$\#(2Q) + h > \#(P + Q)^{\circ}$$

holds then we have: for every nonnegative Laurent polynomial f with monomial support in $2P$ there exists a Laurent polynomial g with monomial support in $2Q$ such that fg is a sum of squares.

Multiplier Bounds

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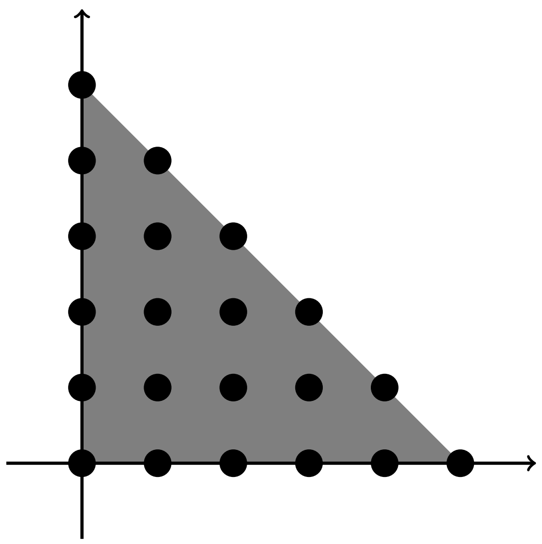

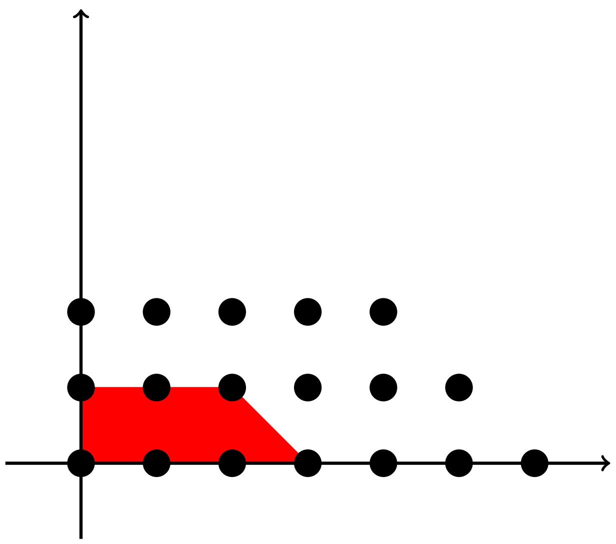

Then $\#(2Q) = 18$ and $h = 3$, while $\#(P + Q)^{\circ} = 20$.

Multiplier Bounds

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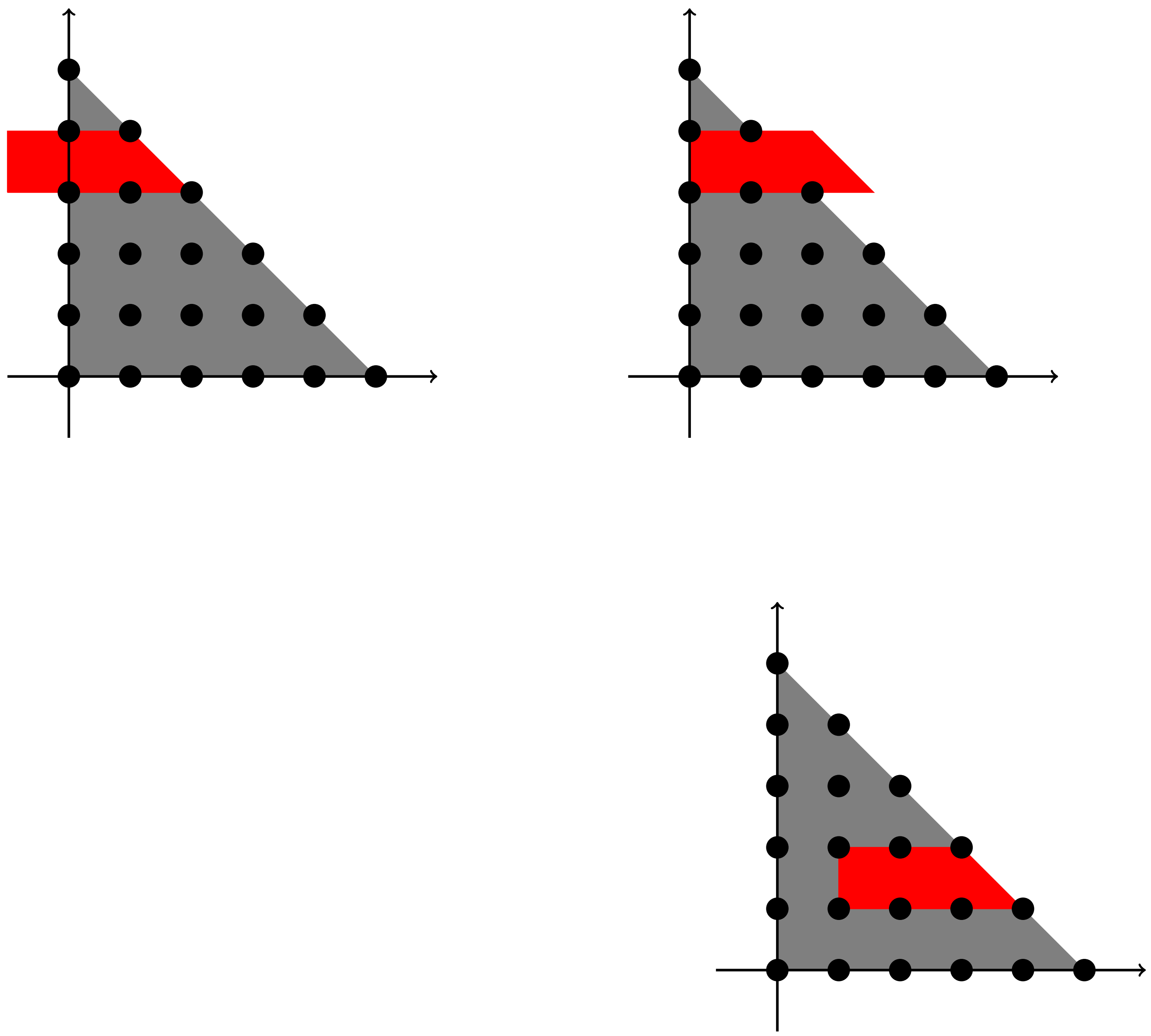

First improvement: 2d=10

Example. *Hilbert's result: for every nonnegative form* $f \in \mathbb{R}[x, y, z]$ *of degree* 10*, there is a sum of squares* g *of degree* 8 *such that* $g f$ *is a sum of squares.*
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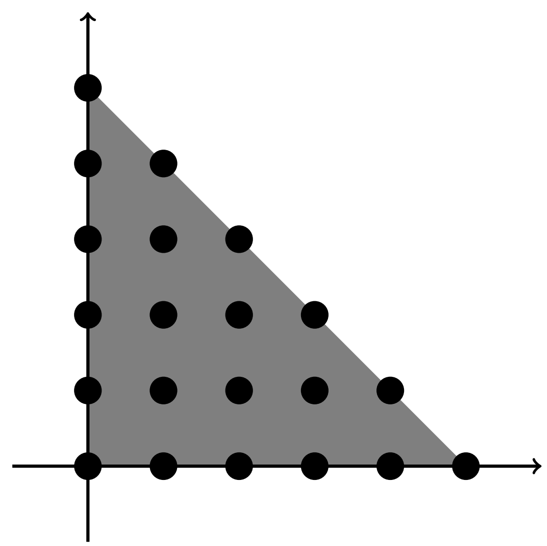
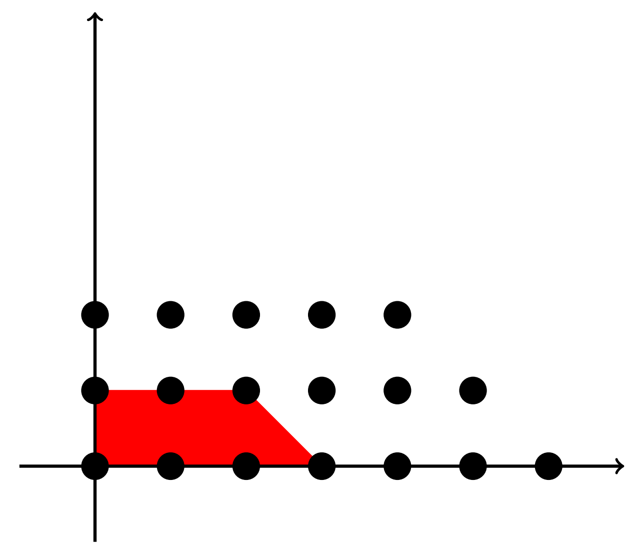




Hilbert vs. Toric Surfaces

For $P = d \cdot \Delta_2$, $Q = (d-2) \cdot \Delta_2$ we have

$$
\begin{aligned}
2Q &= (2d-4) \cdot \Delta_2 \\
\#(2Q) &= \binom{2d-2}{2} \\
h &= 0 \\
(P+Q)^\circ &\sim (2d-5) \cdot \Delta_2 \\
\#\,(P+Q)^\circ &= \binom{2d-3}{2}
\end{aligned}
$$

so that the inequality $\#(2Q) + h > \#\,(P+Q)^\circ$ holds with quite some slack $(2d-3)$.

Hilbert vs. Toric Surfaces

For $P = d \cdot \Delta_2$, $Q = (d-3) \cdot \Delta_2$ we have

$$2Q = (2d-6) \cdot \Delta_2$$

$$\#(2Q) = \binom{2d-4}{2}$$

$$h = 0$$

$$(P+Q)^\circ \sim (2d-6) \cdot \Delta_2$$

$$\#\,(P+Q)^\circ = \binom{2d-4}{2}$$

so that we have equality $\#(2Q) + h = \#\,(P+Q)^\circ$.

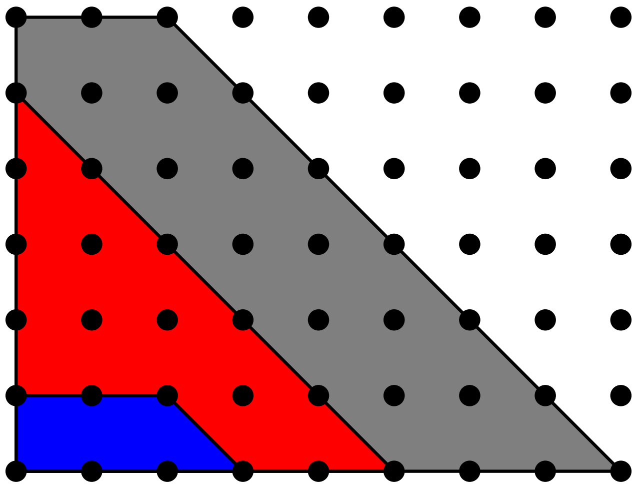
Multiplier Bounds: Aymptotics

Hilbert: Recursively applyting Hilbert's result gives a sum-of-squares multiplier of degree

$$\sum_{j=1}^{d/2}(2d-4j) \sim \frac{d}{2}\cdot 2d - 4\binom{d/2+1}{2} \sim d^2 - 4\cdot\frac{1}{2}\left(\frac{d}{2}\right)^2 \sim \frac{d^2}{2}$$

BSSV: We take roughly $d/3$ steps of size 6 so that the asymptotics is about

$$\sum_{j=1}^{d/3}(2d-6j) \sim \frac{d}{3}\cdot 2d - 6\binom{d/3+1}{2} \sim \frac{2}{3}d^2 - 6\cdot\frac{1}{2}\left(\frac{d}{3}\right)^2 \sim \frac{d^2}{3}$$

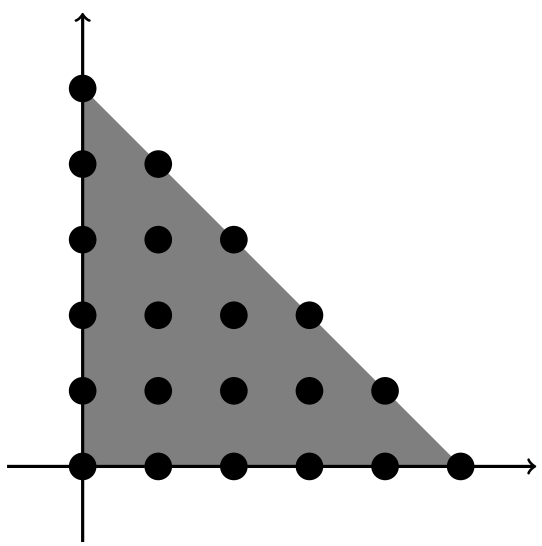


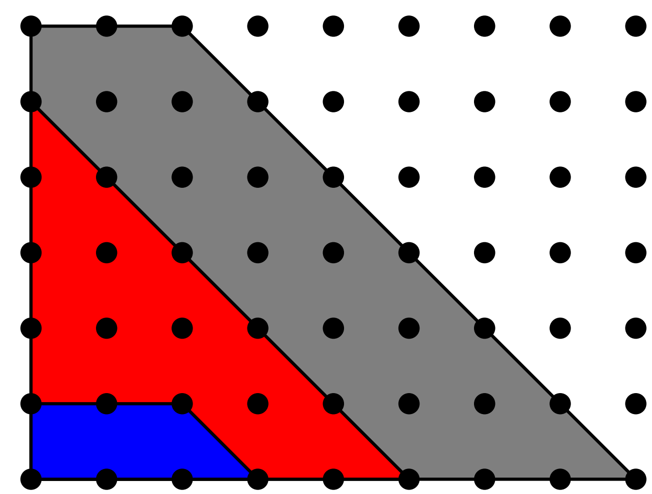
Asymptotics

Let $P = d \cdot \Delta_2$. What is an optimal sequence $(P = Q_0, Q_1, \ldots, Q_r)$ such that

$$\#(2Q_i) + h(Q_{i-1}, Q_i) > \#\left((Q_{i-1} + Q_i)^\circ\right)$$

holds for every pair ($i = 1, \ldots, r$) and such that every nonnegative polynomial in $\mathbb{R}[2Q_r]$ is a sum of squares?





Theorem (Blekherman, Smith, Velasco, 2016). *The lattice polygons Q such that every nonnegative polynomial in $\mathbb{R}[2Q]$ is a sum of squares are the following (up to translations and lattice isomorphisms)*

(1) $\Delta_2 = \text{conv}\{(0,0),(1,0),(0,1)\}$,

(2) $2\Delta_2 = \text{conv}\{(0,0),(2,0),(0,2)\}$,

(3) $Q_{a,b} = \text{conv}\{(0,0),(a,0),(0,1),(b,1)\}$ *(with $a > b$).*