Spectrahedral shadows

Claus Scheiderer

Universität Konstanz

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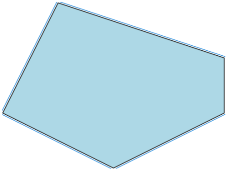
Alex Prestel (1941–2024)

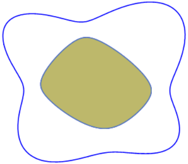
(Photo courtesy of Rudolf Scharlau)

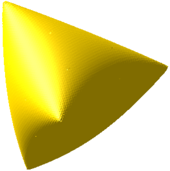
A *spectrahedron* is the solution set of a linear matrix inequality (LMI):

$$K = \left\{ x \in \mathbb{R}^n \colon A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0 \right\}$$

with $A_i \in \mathbb{S}^d = \mathrm{Sym}_d(\mathbb{R})$, some $d \geq 1$. Spectrahedra are closed convex s.a. sets:







$\mathrm{diag}(f_1, \ldots, f_d) \succeq 0$
f_i linear

$$\begin{pmatrix} 3-2x_1 & -x_1 & 2x_2 & -2x_2 \\ -x_1 & 3+2x_1 & -x_2 & -2x_2 \\ 2x_2 & -x_2 & 2-2x_1 & 0 \\ -2x_2 & -2x_2 & 0 & 2+2x_1+x_2 \end{pmatrix} \succeq 0$$

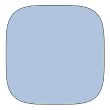
$$\begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} \succeq 0$$

Spectrahedral shadows are the linear images of spectrahedra. They are the solution sets of *lifted LMIs*:

$$K = \left\{ x \in \mathbb{R}^n \colon \exists\, y \in \mathbb{R}^m \; A_0 + \sum_{i=1}^{n} x_i A_i + \sum_{i=1}^{m} y_j B_j \succeq 0 \right\}$$

(with $A_i,\, B_j \in \mathbb{S}^d$, some m, d) and are convex s.a. sets. E.g. $K = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^4 + x_2^4 \le 1\}$ ("tv-screen"):

$$K = \left\{ x \in \mathbb{R}^2 \colon \exists\, u \in \mathbb{R}^2 \begin{pmatrix} 1+u_1 & u_2 & 0 & 0 & 0 & 0 \\ u_2 & 1-u_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1 & x_1 & 0 & 0 \\ 0 & 0 & x_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_2 & x_2 \\ 0 & 0 & 0 & 0 & x_2 & 1 \end{pmatrix} \succeq 0 \right\}$$



Background: Semidefinite programming SDP (optimize linear functions over spectrahedra). Spectrahedral shadows are precisely the feasible sets of SDP.

Spectrahedral shadows are convex and semialgebraic. What else?

Nemirovskii (ICM Madrid 2006): *Is every convex semialgebraic set a shadow?*

Meanwhile much better understood:

- (Helton-Nie 2009/2010) Sufficient geometric conditions for convex $K \subseteq \mathbb{R}^n$ to be a shadow (e.g. K compact with non-singular boundary of strict positive curvature)
- (Sch. 2017) There exist closed convex s.a. non-shadows. Prominent example:

$$P_{n,2d} = \left\{ f \in \mathbb{R}[x_1, \dots, x_n] \colon f \text{ homogeneous},\ \deg(f) = 2d,\ f \geq 0 \text{ on } \mathbb{R}^n \right\}$$

  is a non-shadow precisely if $P_{n,2d} \neq \Sigma_{n,2d}$ (ie iff $2d \geq 6$ and $n \geq 3$, or $(n, 2d) = (4, 4)$, Hilbert 1888)
- (Bodirsky-Kummer-Thom 2024) $C_n = \{A \in \mathbb{S}^n \colon \forall\, x \in \mathbb{R}^n_+\ x^t Ax \geq 0\}$ (cone of copositive symmetric $n \times n$-matrices) is a non-shadow for $n \geq 5$. Had been a well-known open question for years.
- Many more explicit results, e.g. Hess-Goel-Kuhlmann 2025: Filtration $\Sigma_{n,2d} \subsetneq C_1 \subsetneq \cdots \subsetneq C_k = P_{n,2d}$ by intermediate cones C_i; none of the C_i is a shadow.

(more)

- (Sch. 2017) The closed convex hull of any one-dimensional s.a. set in $\mathbb{R}^n$ is a shadow. But any s.a. set of dimension ≥ 2 can be imbedded in $\mathbb{R}^N$ (for some N) in such a way that its closed convex hull in $\mathbb{R}^N$ is a non-shadow.
- Several equivalent (quite different) characterizations for being a shadow are known. However, none is easy to check in general.

Many natural questions still open, e.g.:

- Every convex s.a. set in $\mathbb{R}^2$ is a shadow. How about sets in $\mathbb{R}^3$? in $\mathbb{R}^4$? ... Smallest known example of a non-shadow lives in $\mathbb{R}^{11}$!
- Do there exist non-shadows with smooth boundary? All non-shadows constructed so far have singular boundary.

To perform SDP over a shadow K one uses a lifted LMI representation

$$K = \{x \in \mathbb{R}^n \colon \exists\, y \in \mathbb{R}^m \; M(x,y) \succeq 0\}$$

Small matrix size is preferable: A block-diagonal LMI with many small blocks performs MUCH faster than one large-size LMI with dense entries.

Definition: (Averkov 2019) For $K \subseteq \mathbb{R}^n$ convex let $\operatorname{sxdeg}(K)$ (*semidefinite extension degree* of K) be the smallest $d \geq 1$ such that K has a lifted LMI representation

$$K = \{x \in \mathbb{R}^n \colon \exists\, y \in \mathbb{R}^m \; M_1(x,y) \succeq 0, \dots, M_r(x,y) \succeq 0\}$$

with linear matrix polynomials $M_i(x,y)$ of size at most $d \times d$.

- $\operatorname{sxdeg}(K) = 1 \Leftrightarrow K$ is a polyhedron
- $\operatorname{sxdeg}(K) \leq 2 \Leftrightarrow K$ is *second-order cone (soc)* representable
- $\operatorname{sxdeg}(K) < \infty \Leftrightarrow K$ is a shadow

$\operatorname{sxdeg}(K)$ defines a hierarchy for the intrinsic complexity of SDP over the set K

Lower bounds: (Averkov) General criterion on K (of mixed geometric / combinatorial character) that guarantees $\mathrm{sxdeg}(K) \geq d$. Interesting cases e.g.:

- $\mathrm{sxdeg}(\mathbb{S}_+^d) = d$ (and not less) (Fawzi 2019 for $d = 3$, Averkov 2019 in general)
- $\mathrm{sxdeg}(\Sigma_{n,2d}) = \mathrm{sxdeg}(\Sigma_{n,2d}^*) = \binom{n+d-1}{d} = \dim \mathbb{R}[x_1, \ldots, x_n]_d$ (and not less). Note: $\Sigma_{n,2d}^*$ is a spectrahedral cone, naturally described by an LMI of size $\binom{n+d-1}{d}$.

Upper bound:

- (Sch. 2024) If $K \subseteq \mathbb{R}^n$ is the closed convex hull of a one-dimensional s.a. set then $\mathrm{sxdeg}(K) \leq \lfloor \frac{n}{2} \rfloor + 1$.
- In particular, $\mathrm{sxdeg}(K) \leq 2$ for every closed convex set $K \subseteq \mathbb{R}^2$

Bound is sharp: $K = \overline{\mathrm{conv}\{(t, t^2, \ldots, t^n)\colon t \in \mathbb{R}\}} \subseteq \mathbb{R}^n$, then $\mathrm{sxdeg}(K) \geq \lfloor \frac{n}{2} \rfloor + 1$ by Averkov's criterion. Explicit LMI representation of this size:

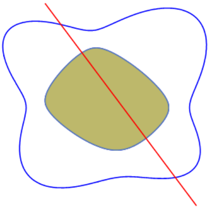
$$K = \left\{ x \in \mathbb{R}^n \colon \begin{pmatrix} 1 & x_1 & \cdots & x_k \\ x_1 & x_2 & \cdots & x_{k+1} \\ \vdots & \vdots & & \vdots \\ x_k & x_{k+1} & \cdots & x_{2k} \end{pmatrix} \succeq 0 \right\}, \quad k = \left\lfloor \frac{n}{2} \right\rfloor$$

Again: Equivalent characterizations for $\mathrm{sxdeg}(K) \le d$ are known. But in general, sxdeg is not well understood. Some open questions:

- Let $K \subseteq \mathbb{R}^3$ be closed convex s.a. Then is $\mathrm{sxdeg}(K) \le 2$? No counter-example known. But not even known whether $\mathrm{sxdeg}(K) < \infty$!
- More generally, let n be arbitrary, let $K \subseteq \mathbb{R}^n$ be a shadow. Then is $\mathrm{sxdeg}(K) \le \lfloor \frac{n}{2} \rfloor + 1$? (the bound for convex hulls of curves) No counter-example known.

A form $f = f(x_1, \ldots, x_n)$ is *hyperbolic* wrt $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and if for every $u \in \mathbb{R}^n$, the polynomial $f(te+u) \in \mathbb{R}[t]$ is real-rooted. Put $U_e(f)$ = connected component of e in $\{x \in \mathbb{R}^n \colon f(x) \neq 0\}$. The (closed) *hyperbolicity cone* of f is $C_e(f) := \overline{U_e(f)}$. This is a closed convex cone (Gårding 1959).

$$f = \det \begin{pmatrix} 3z-2x & -x & 2y & -2y \\ -x & 3z+2x & -y & -2y \\ 2y & -y & 2z-2x & 0 \\ -2y & -2y & 0 & 2z+2x+y \end{pmatrix}, \; z = 1$$



Standard example: Symmetric linear matrix polynomial $A(x) = \sum_{i=1}^n x_i A_i$ (with $A_i \in \mathbb{S}^d$). If $e \in \mathbb{R}^n$ satisfies $A(e) \succ 0$ then $f(x) := \det A(x)$ is hyperbolic wrt e, with hyperbolicity cone

$$C_e(f) = \{x \in \mathbb{R}^n \colon A(x) \succeq 0\}$$

So $C_e(f)$ is a spectrahedral cone in this case.

In $n = 3$ variables this gives all hyperbolic forms: Former Lax Conjecture (1958), proved by Helton-Vinnikov (2007).

For $n > 3$ there are much more hyperbolic forms. But perhaps not more hyperbolicity cones?

Generalized Lax Conjecture (GLC): *For every n and every hyperbolic form $f \in \mathbb{R}[x_1, \ldots, x_n]$, the hyperbolicity cone $C_e(f)$ is a spectrahedron, ie described by a linear matrix inequality.*

Essentially wide open. Not even known if the *Weak GLC* holds, which predicts that, at least, $C_e(f)$ is always a spectrahedral shadow.

The Weak GLC has been proved by Netzer-Sanyal (2015) in the case where $C_e(f)$ has smooth boundary.

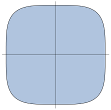
Theorem 1: *Let f be a hyperbolic form such that $C_e(f)$ has smooth boundary. Then $C_e(f)$ is soc-representable, ie* $\operatorname{sxdeg} C_e(f) = 2$.

Is a consequence of a more general result:

Theorem 2: *Let $K \subseteq \mathbb{R}^n$ be a compact convex s.a. set such that ∂K is everywhere smooth of strict positive curvature. Then* $\operatorname{sxdeg}(K) = 2$.

Let $K \subseteq \mathbb{R}^n$ be closed convex. A point $u \in \partial K$ is a *smooth boundary point* of K if $\exists\, g \in \mathbb{R}[x]$ with $\nabla g(u) \neq 0$ and $K \cap U = \{g \geq 0\} \cap U$ for some neighborhood U of u. Moreover, ∂K has *strict positive curvature* at u if $x^t \nabla^2 g(u) x < 0$ for all $0 \neq x \in \mathbb{R}^n$ with $\langle x, \nabla g(u) \rangle = 0$. E.g.

$$K = \{x \in \mathbb{R}^2 \colon x_1^4 + x_2^4 \leq 1\}$$



has smooth boundary and strict positive curvature in all points except $(\pm 1, 0)$, $(0, \pm 1)$.

Theorem 1 is reduced to Theorem 2 by intersecting the hyperbolicity cone with a suitable affine hyperplane.

Proof of Theorem 2 uses concept of *tensor evaluation*: Let $K \subseteq \mathbb{R}^n$ be convex with smooth boundary. Given a real closed field $R \supseteq \mathbb{R}$ and $\eta \in \partial K_R$, let $t_\eta \in R[x]$ be the positive tangent to ∂K_R at η. For $\xi \in \partial K_R$ consider $t_\eta^{\otimes}(\xi) \in R \otimes R$, the image of t_η under

$$R[x] \;=\; \mathbb{R}[x] \otimes R \xrightarrow{\xi \otimes 1} R \otimes R$$

Showing $\operatorname{sxdeg}(K) \le 2$ means to show that $t_\eta^{\otimes}(\xi) = \sum_i \big(a_i \otimes b_i - c_i \otimes d_i\big)^2$ in $R \otimes R$ ($\forall$ choices of R, ξ, η).

Reduce to case where ξ, η specialize to same $\mathbb{R}$-point $u \in \partial K_{\mathbb{R}}$. Then use local Taylor expansion of f at u.

Algebraic fact used: If A is a f.g. smooth $\mathbb{R}$-algebra of dimension n for which $\Omega_{A/\mathbb{R}}$ is free, and if $I = \ker(A \otimes A \xrightarrow{mult} A)$, then

$$(A \otimes A)/I^d \;\cong\; A[y_1, \dots, y_n]/\langle y_1, \dots, y_n\rangle^d \quad \forall\, d \ge 1$$

A related technique is used for second main result:

A form $f \in \mathbb{R}[x_1, \dots, x_n]$ is *convex* if $f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$ for all $x, y \in \mathbb{R}^n$. Every convex form f with $\deg(f) \ge 2$ is ≥ 0 on $\mathbb{R}^n$. Parrilo (2007) asked: Is f necessarily sos?

Theorem: (Blekherman 2009) *Let $2d \ge 4$ be fixed. Then for $n \gg 0$ there exists a convex form of degree $2d$ in n variables that is not sos.*

Asymptotically for $n \to \infty$, $vol(K_{n,2d})$ grows stronger than $vol(\Sigma_{n,2d})$.

Saunderson (2022): First explicit example of such a form, of degree 4 in 272 variables!

Theorem 3: *Assume that $f \in \mathbb{R}[x_1, \dots, x_n]$ is a convex form that is not sos. Then the upper hull of graph(f),*

$$U(f) \ = \ \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \colon x_0 \ge f(x_1, \dots, x_n)\right\},$$

is (convex and) not a shadow.

Corollary: *There exist closed convex non-shadows with smooth boundary.*

Question: What is the smallest $n \ge 3$ such that there exists a convex form in n variables that is not sos?

Congratulations DDG, and Many Happy Returns!