Sylvester double sums when there are multiplicities and symmetric Hermite interpolation

Marie-Françoise Roy

based on a work in common with

Aviva Szpirglas

IRMAR, Université Rennes, France

DDG40, Banyuls,
August 7 2025

What is it about ?

(P and Q two univariate polynomials)

- unify two appproaches of the euclidean remainder sequence of P and Q
- subresultants starting from Euler, famous in computer algebra, defined through minors containing the coefficients of P and Q (so, no denominators), proportional to the euclidean remainder sequence of P and Q
- Sylvester double sums, introduced by Sylvester in 1840, less famous, defined through symmetric expressions of the roots of P and Q, also proportional to the euclidean remainder sequence of P and Q (classically: only when there are no multiple roots)
- the equality of two expressions of the resultant in terms of the determinant of Sylvester matrix and in terms of the roots of P and Q is a prototype.

Resultant

$\mathbf{P} = \{x_1, \ldots, x_p\}$, $\mathbf{Q} = \{y_1, \ldots, y_q\}$ $(q < p)$

$\Pi(\mathbf{P}, \mathbf{Q}) = \prod_{a \in \mathbf{P}, b \in \mathbf{Q}} (a - b) = (-1)^{pq} \prod_{a \in \mathbf{P}, b \in \mathbf{Q}} (b - a)$

$P(U) = \Pi(U, \mathbf{P})$, $Q(U) = \Pi(U, \mathbf{Q})$

Proposition (two expressions for the resultant)

$$\Pi(\mathbf{P}, \mathbf{Q}) = \varepsilon \mathrm{det} \mathrm{SH}(P, Q)$$

where $\mathrm{SH}(P, Q)$ *, the Sylvester-Habicht matrix, is the* $p + q$ *square matrix with rows* $X^{q-1} P \ldots, P, Q, \ldots X^{p-1} Q$ *in the basis* $X^{p+q-1}, \ldots, 1$ *and* ε *is a sign.*

By induction on the length of the remainder sequence, going from (P, Q) to (Q, R), where $R = -\mathrm{Rem}(P, Q)$

Resultant

By induction on the length of the remainder sequence, going from (P, Q) to (Q, R), where $R/c_r = \Pi(U, \mathbf{R})$

$$\begin{aligned}\Pi(\mathbf{P}, \mathbf{Q}) &= (-1)^{pq} \prod_{b \in \mathbf{Q}} P(b) = (-1)^{q(p-q)} \prod_{b \in \mathbf{Q}} R(b) \\ &= (-1)^{q(p-q)} c_r^q \Pi(\mathbf{Q}, \mathbf{R})\end{aligned}$$

Key: $P = CQ - R \implies P(b) = -R(b), b \in \mathbf{Q}$

$$\mathrm{detSH}(P, Q) = \varepsilon_{p-q} c_r^q \mathrm{detSH}(Q, R/c_r)$$

$\varepsilon_i = (-1)^{i(i-1)/2}$.

Key: the row of coefficients of $-R$ is obtained by substracting to the row of coefficients of P a linear combination of rows of coefficents of $X^{p-q}Q, \ldots, Q$, which does not change the determinant. Needed to reorder the rows.

What we want to prove and our method

(P and Q two polynomials, $R = -\mathrm{Rem}(P, Q)$)

- Already known: Sylvester double sums, simple roots case, are equal (up to a constant) to subresultants;
- Our aim: define Sylvester double sums when there are multiplicities.
- Our main result: Sylvester double sums are equal (up to a sign) to subresultants in the general case.
- Our proof: by induction on the length of the remainder sequence, using the relationship between the values of both quantities for (P, Q) and for $(Q, R = -\mathrm{Rem}(P, Q))$.
- This method of proof plays a key role in many proofs in real root counting (Sturm theorem, structure theorem of subresultants, Bezoutians) [BPR]

Sylvester double sums, simple roots case

$\mathbf{P} = \{x_1, \dots, x_p\}$, $\mathbf{Q} = \{y_1, \dots, y_q\}$ $(q < p)$
$\mathbf{K} \subset_k \mathbf{P}, \mathbf{L} \subset_\ell \mathbf{Q}$, $\Pi(\mathbf{K}, \mathbf{L}) = \prod_{\substack{a \in \mathbf{K} \\ b \in \mathbf{L}}} (a - b)$.

$P(U) = \Pi(U, \mathbf{P})$, $Q(U) = \Pi(U, \mathbf{Q})$

Definition (classical Sylvester double sums)

$$\mathrm{Sylv}^{k,\ell}(P, Q)(U) = \sum_{\substack{\mathbf{K} \subset_k \mathbf{P} \\ \mathbf{L} \subset_\ell \mathbf{Q}}} \Pi(U, \mathbf{K}) \Pi(U, \mathbf{L}) \frac{\Pi(\mathbf{K}, \mathbf{L}) \Pi(\mathbf{P} \setminus \mathbf{K}, \mathbf{Q} \setminus \mathbf{L})}{\Pi(\mathbf{K}, \mathbf{P} \setminus \mathbf{K}) \Pi(\mathbf{L}, \mathbf{Q} \setminus \mathbf{L})}$$

Definition non sensical when there are multiple roots.

Sylvester motivation

- Hard to guess. His papers [S1840] and [S1853] are not written in modern mathematical terms.
- Double sums are symmetric expression of the roots.
- Connection with gcd.

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = \sum_{\substack{\mathbf{K}\subset_k \mathbf{P}\\ \mathbf{L}\subset_\ell \mathbf{Q}}} \Pi(U,\mathbf{K})\Pi(U,\mathbf{L})\frac{\Pi(\mathbf{K},\mathbf{L})\Pi(\mathbf{P}\setminus\mathbf{K},\mathbf{Q}\setminus\mathbf{L})}{\Pi(\mathbf{K},\mathbf{P}\setminus\mathbf{K})\Pi(\mathbf{L},\mathbf{Q}\setminus\mathbf{L})}$$

If $G = \mathrm{gcd}(P,Q)$ has roots $\{z_1,\ldots,z_g\}$,

- If $j = k+\ell < g$, $\mathrm{Sylv}^{k,\ell}(P,Q)(U) = 0$
- If $j = k+\ell = g$, $\mathrm{Sylv}^{k,\ell}(P,Q)(U)$ proportional to G.

Our motivation: modern correct proofs, with formulas for special cases (multiple roots).

Sylvester double sums, another expression

Vandermonde vector $v_i(U) = \left[U^{j-1}\right]_{0 \le j \le i-1}$.
The Vandermonde determinant $V(\mathbf{T})$ is the determinant of the Vandermonde matrix $\mathbf{T} = (v_i(T_1), \dots, v_i(T_i))$ (ordered list)
$\mathbf{L} \| \mathbf{K}$: ordered set obtained by concatenation
$V(\mathbf{L}\|\mathbf{K}) = V(\mathbf{K}) \Pi(\mathbf{K}, \mathbf{L}) V(\mathbf{L})$
$\sigma_{\mathbf{K}}$: signature of the permutation $\mathbf{P} \mapsto \mathbf{K} \| (\mathbf{P} \setminus \mathbf{K})$
$\sigma_{\mathbf{L}}$: signature of the permutation $\mathbf{Q} \mapsto \mathbf{L} \| (\mathbf{Q} \setminus \mathbf{L})$
New expression of the double sums.

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = \sum_{\substack{\mathbf{K} \subset_k \mathbf{P} \\ \mathbf{L} \subset_\ell \mathbf{Q}}} \sigma_{\mathbf{K}} \sigma_{\mathbf{L}} \frac{V(\mathbf{Q} \setminus \mathbf{L} \| \mathbf{P} \setminus \mathbf{K}) V(\mathbf{L} \| \mathbf{K} \| U)}{V(\mathbf{P}) V(\mathbf{Q})}.$$

New definition also does not make sense when there are multiple roots.

Generalized Vandermonde: an example

Two rooots x_1 (double) and x_2 (triple),
$x_{1,j} = (x_1, j)$, $j = 0, 1$, $x_{2,j} = (x_2, j)$, $j = 0, 1, 2$

$$\mathbf{P} = \{x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}, x_{2,2}\}$$

$$V(X_{\mathbf{P}}) = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ X_{1,0} & X_{1,1} & X_{2,0} & X_{2,1} & X_{2,2} \\ X_{1,0}^2 & X_{1,1}^2 & X_{2,0}^2 & X_{2,1}^2 & X_{2,2}^2 \\ X_{1,0}^3 & X_{1,1}^3 & X_{2,0}^3 & X_{2,1}^3 & X_{2,2}^3 \\ X_{1,0}^4 & X_{1,1}^4 & X_{2,0}^4 & X_{2,1}^4 & X_{2,2}^4 \end{pmatrix}$$

$$V[\mathbf{P}] = \partial^{[\mathbf{P}]}(V(X_{\mathbf{P}}))(\mathbf{P}) = \det \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ x_1 & 1 & x_2 & 1 & 0 \\ x_1^2 & 2x_1 & x_2^2 & 2x_2 & 1 \\ x_1^3 & 3x_1^2 & x_2^3 & 3x_2^2 & 3x_2 \\ x_1^4 & 4x_1^3 & x_2^4 & 4x_2^3 & 6x_2^2 \end{pmatrix}$$

$$= (x_2 - x_1)^6 \neq 0$$

Generalized Vandermonde

$$\mathbf{P} = \{x_{1,0}, \ldots, x_{1,\mu_1-1}, \ldots, x_{k,0}, \ldots, x_{k,\mu_k-1}\},$$

$x_{i,j} = (x_i, j)$, $x_i \neq x_{i'}$ for $i \neq i'$, $\sum_{i=1}^{k} \mu_i = p$.

$$\mathbf{Q} = \{y_{1,0}, \ldots, y_{1,\nu_1-1}, \ldots, y_{\ell,0}, \ldots, y_{\ell,\nu_\ell-1}\},$$

$y_{i,j} = (y_i, j)$, $y_i \neq y_{i'}$ for $i \neq i'$, $\sum_{i=1}^{\ell} \nu_i = q$.

$X_{\mathbf{P}} = \{X_{1,0}, \ldots, X_{1,\mu_1-1}, \ldots, X_{k,0}, \ldots, X_{k,\mu_k-1}\}$,
$Y_{\mathbf{Q}} = \{Y_{1,0}, \ldots, Y_{1,\nu_1-1}, \ldots, Y_{k,0}, \ldots, Y_{k,\nu_k-1}\}$

Generalized Vandermonde

For $f \in K[\mathbf{V}][U]$,

$$\frac{\partial^{[i]} f}{\partial U^i} = \frac{1}{i!}\frac{\partial^i f}{\partial U^i}$$

For $f \in K[X_{\mathbf{P}}]$ and $\mathbf{K} \subset_k \mathbf{P}$, with $X_{\mathbf{K}} = \left\{X_{i,j} | x_{i,j} \in \mathbf{K}\right\}$, define $\partial^{[\mathbf{K}]}(f)$ by

$$\partial^{[\emptyset]} f = f, \qquad \text{for } \mathbf{K} = \mathbf{K}' \| \left\{x_{i,j}\right\}, \partial^{[\mathbf{K}]} f = \frac{\partial^{[j]} \partial^{[\mathbf{K}']} f}{\partial X_{i,j}^{j}}$$

Definition (Generalized Vandermonde determinant)

$$V[\mathbf{P}] = \partial^{[\mathbf{P}]}(V(X_{\mathbf{P}}))(\mathbf{P}).$$

$$V[\mathbf{P}] = \prod_{1 \le i < j \le k} (x_j - x_i)^{\mu_i \mu_j}$$

General Sylvester double sums

Definitions did not make sense when there are mulltiple roots.

$$\frac{0}{0}$$

use L'Hôpital's Rule: derivate in a relevant way numerator and denominator.

$$V[\mathbf{L}||\mathbf{K}||U) = \partial^{[\mathbf{K}]}\partial^{[\mathbf{L}]}V(Y_{\mathbf{L}}||X_{\mathbf{K}}||U)(\mathbf{L}||\mathbf{K})$$

(derivation with respect to $X_{\mathbf{P}}$, $Y_{\mathbf{Q}}$ not with respect to U)

Definition (Sylvester double sums)

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = \sum_{\substack{\mathbf{K}\subset_k \mathbf{P} \\ \mathbf{L}\subset_\ell \mathbf{Q}}} \sigma_{\mathbf{K}}\sigma_{\mathbf{L}} \frac{V[\mathbf{Q}\setminus\mathbf{L}||\mathbf{P}\setminus\mathbf{K}]\,V[\mathbf{L}||\mathbf{K}||U)}{V[\mathbf{P}]\,V[\mathbf{Q}]}$$

Subresultants, P, Q non monic

Subresultants are defined by minors of the Sylvester-Habicht matrix and are proportional to the polynomials in the remainder sequence.
$R = -\mathrm{Rem}(P, Q)$, $\varepsilon_i = (-1)^{i(i-1)/2}$. The following is well known

Proposition (induction for subresultants)

1. $\mathrm{Sres}_{p-1}(P, Q)(U) = Q(U)$
2. $\mathrm{Sres}_j(P, Q)(U) = 0 \quad q < j < p-1$
3. $\mathrm{Sres}_q(P, Q)(U) = \varepsilon_{p-q}\, lc(Q)^{p-q-1} Q(U)$
4. $\mathrm{Sres}_j(P, Q)(U) = \varepsilon_{p-q}\, lc(Q)^{p-r}\mathrm{Sres}_j(Q, R)(U)$ *if* $j < q$, $R \neq 0$
5. $\mathrm{Sres}_j(P, Q)(U) = 0$ *if* $j < q$, $R = 0$

Our aim for Sylvester double sums, P, Q non monic

Definition in the non monic case:

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = \mathrm{lc}(P)^{q-j}\mathrm{lc}(Q)^{p-j}\mathrm{Sylv}^{k,\ell}\left(\frac{P}{\mathrm{lc}(P)},\frac{Q}{\mathrm{lc}(Q)}\right)(U)$$

Proposition (induction for double sums)

1. $\mathrm{Sylv}^{p-1,0}(P,Q,U) = (-1)^{p-1}\mathrm{lc}(P)^{q-p+1}Q(U)$
2. $\mathrm{Sylv}^{j,0}(P,Q)(U) = 0$, $q < j < p-1$
3. $\mathrm{Sylv}^{q,0}(P,Q)(U) = (-1)^{q(p-q)}\mathrm{lc}(Q)^{p-q-1}Q(U)$
4. $\mathrm{Sylv}^{j,0}(P,Q)(U) = (-1)^{q(p-q)}\mathrm{lc}(Q)^{p-r}\mathrm{Sylv}^{j,0}(Q,R)(U)$ *if* $j < q$, $R \neq 0$
5. $\mathrm{Sylv}^{j,0}(P,Q)(U) = 0$ *if* $j < q$, $R = 0$

What we want to do

- Prove that $\mathrm{Sylv}^{k,\ell}(P,Q)(U)$ is proportional to $\mathrm{Sylv}^{j,0}(P,Q)(U)$, $j = k + \ell$
- Using the two propositions (induction for subresultants, induction for double sums) to prove that $\mathrm{Sres}_j(P,Q)(U)$ and $\mathrm{Sylv}^{j,0}(P,Q)(U)$ coincide up to sign.
- We would like to use interpolation to prove equalities in the induction for double sums but there are $\binom{p}{k}$ subsets of **P** or cardinal k ! Solution: add variables !

Classical Hermite Interpolation

U: one indeterminate

Proposition

Given an ordered list

$$\mathbf{q} = (q_{1,0}, \ldots, q_{1,\mu_1-1}, \ldots, q_{m,0}, \ldots, q_{m,\mu_m-1})$$

of p numbers, there is one and only one polynomial of degree at most $p-1$ satisfying the property

$$\text{for all } 1 \leq i \leq m, \ \text{for all } 0 \leq j < \mu_i, \ Q^{[j]}(x_i) \quad = \quad q_{i,j}.$$

(generalizes Classical Lagrange interpolation)

Classical Hermite Interpolation

U: one indeterminate . Hermite interpolation basis (in an example)
Two rooots x_1 (double) and x_2 (triple),

$$\mathbf{P} = \{x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}, x_{2,2}\}$$

$$\mathcal{V}[\mathbf{P}\|U) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ x_1 & 1 & x_2 & 1 & 0 & U \\ x_1^2 & 2x_1 & x_2^2 & 2x_2 & 1 & U^2 \\ x_1^3 & 3x_1^2 & x_2^3 & 3x_2^2 & 3x_2 & U^3 \\ x_1^4 & 4x_1^3 & x_2^4 & 4x_2^3 & 6x_2^2 & U^4 \end{pmatrix}$$

them remove one of the five first columns, compute the determinant and divide by $V[\mathbf{P}]$: Hermite interpolation basis of polynomials of degree ≤ 4.

Multisymmetric Hermite Interpolation

U: a block of indeterminates of cardinality $p - k$

Proposition

$$\mathcal{B}_{\mathbf{P},k} = \left\{ \frac{V[\mathbf{K}||\mathbf{U})}{V[\mathbf{P}]V(\mathbf{U})} \right\}$$

indexed by $\mathbf{K} \subset_k \mathbf{P}$ *is a linear basis of the set of symmetric polynomials in* $\mathbf{U}$ *of multidegree at most* $k, \ldots, k$*.*

If g *is such a symmetric polynomial in* $\mathbf{U}$ *,*

$$g = \sum_{\mathbf{K} \subset_k \mathbf{P}} \sigma_{\mathbf{K}} \partial^{[\mathbf{P} \setminus \mathbf{K}]}(g(X_{\mathbf{P}\setminus\mathbf{K}})V(X_{\mathbf{P}\setminus\mathbf{K}}))(\mathbf{P} \setminus \mathbf{K}) \frac{V[\mathbf{K}||\mathbf{U})}{V[\mathbf{P}]V(\mathbf{U})}$$

(our definition, generalizing Multisymmetric Lagrange interpolation used by T. Krick, A. Szanto, M. Valdetarro)

As a corollary: classical Hermite Interpolation.

Multi Sylvester double sums

Add variables to be able to prove equalities by interpolation: idea borrowed from T. Krick, A. Szanto, and M. Valdetarro (2016)

Definition (Multi Sylvester double sums)

P and Q: monic polynomials, $j = k + \ell$, $\#\mathbf{U} = p - j$.

$$\mathrm{MSylv}^{k,\ell}(P,Q)(\mathbf{U}) = \sum_{\substack{\mathbf{K}\subset_k \mathbf{P}\\ \mathbf{L}\subset_\ell \mathbf{Q}}} \sigma_{\mathbf{K}}\sigma_{\mathbf{L}} \frac{V[\mathbf{Q}\setminus\mathbf{L}\|\mathbf{P}\setminus\mathbf{K}]\,V[\mathbf{L}\|\mathbf{K}\|\mathbf{U})}{V[\mathbf{P}]\,V[\mathbf{Q}]\,V(\mathbf{U})} \quad (1)$$

(derivation with respect to $X_{\mathbf{P}}$, $Y_{\mathbf{Q}}$ not with respect to $\mathbf{U}$)

Proposition

$\mathrm{Sylv}^{k,\ell}(P,Q)(U)$ *is the coefficient of* $\prod_{U'\in\mathbf{U}'} U'^j$ *in*

$\mathrm{MSylv}^{k,\ell}(P,Q)(U\|\mathbf{U}')$, $\#\mathbf{U}' = p - j - 1$

Multi Sylvester double sums for $j \geq q$

P and Q two monic polynomials

Proposition

if $\ell \leq q \leq k + \ell = j < p$ then

$$\binom{q}{\ell} \mathrm{MSylv}^{j,0}(P, Q, \mathbf{U}) = (-1)^{\ell(p-j)} \mathrm{MSylv}^{k,\ell}(P, Q, \mathbf{U})$$

Proof rather easy using interpolation!
By taking a relevant coefficient,

Corollary

P and Q two monic polynomials; if $\ell \leq q \leq k + \ell = j < p$ then

$$\binom{q}{\ell} \mathrm{Sylv}^{j,0}(P, Q, \mathbf{U}) = (-1)^{\ell(p-j)} \mathrm{Sylv}^{k,\ell}(P, Q, \mathbf{U})$$

Proposition

If $q \leq j \leq p - 1$

$$\mathrm{MSylv}^{j,0}(P,Q)(\mathbf{U}) = (-1)^{j(p-j)} lc(P)^{q-j} \prod_{U \in \mathbf{U}} Q(U)$$

Proof "Easy", using multisymmetric Hermite interpolation

$$\prod_{U \in \mathbf{U}} Q(U) = \sum_{\mathbf{K} \subset_j \mathbf{P}} \sigma_{\mathbf{K}} \partial^{[\mathbf{P} \setminus \mathbf{K}]} \left(V(X_{\mathbf{P} \setminus \mathbf{K}}) \prod_{X \in X_{\mathbf{P} \setminus \mathbf{K}}} Q(X) \right) (\mathbf{P} \setminus \mathbf{K}) \frac{V[\mathbf{K} || \mathbf{U})}{V[\mathbf{P}] V(\mathbf{U})}$$

By taking a relevant coefficient,

Corollary

1. $\mathrm{Sylv}^{p-1,0}(P,Q,U) = (-1)^{p-1} \mathrm{lc}(P)^{q-p+1} Q(U)$
2. $\mathrm{Sylv}^{j,0}(P,Q,U) = 0,\ q < j < p-1,$
3. $\mathrm{Sylv}^{q,0}(P,Q,U) = (-1)^{q(p-q)} \mathrm{lc}(Q)^{p-q-1} Q$

Multi Sylvester double sums for $\mathbf{0 \leq j < q}$

Proposition

If $k \in \mathbb{N}$, $\ell \in \mathbb{N}$, $k + \ell = j < q$ and $\mathbf{U}$ a set of $p - j$ indeterminates,

$$\mathrm{MSylv}^{k,\ell}(P,Q)(\mathbf{U}) = \binom{j}{k} (-1)^{\ell(p-j)} \mathrm{MSylv}^{j,0}(P,Q)(\mathbf{U}).$$

The proof is complicated and uses the Exchange Lemma from T. Krick, A. Szanto, and M. Valdetarro (2016).
By taking a relevant coefficient,

Corollary

If $k \in \mathbb{N}$, $\ell \in \mathbb{N}$, $k + \ell = j < q$,

$$\mathrm{Sylv}^{k,\ell}(P,Q)(U) = \binom{j}{k} (-1)^{\ell(p-j)} \mathrm{Sylv}^{j,0}(P,Q)(U).$$

Sylvester double sums and remainder

It remains to prove for P and Q non monic

Proposition

- $\mathrm{Sylv}^{j,0}(P,Q)(U) = (-1)^{q(p-q)}\mathrm{lc}(Q)^{p-r}\mathrm{Sylv}^{j,0}(Q,R)(U)$ *if* $j < q$, $R \neq 0$
- $\mathrm{Sylv}^{j,0}(P,Q)(U) = 0$ *if* $j < q$, $R = 0$

The proof uses as essential ingredient

Lemma

$R = -\mathrm{Rem}(P,Q)$. *For every* y_i *such that* $Q(y_i) = 0$, $0 \leq j \leq \nu_i - 1$,

$$P^{[j]}(y_i) = -R^{[j]}(y_i)$$

And also the proportionnality between $\mathrm{Sylv}^{j,0}(Q,R)(U)$ and $\mathrm{Sylv}^{0,j}(Q,R)(U)$.

Conclusion

- We introduced general Sylvester double sums making sense when there are multiplicities using Generalized Vandermonde determinants
- We proved that $\mathrm{Sylv}^{k,\ell}(P,Q)(U)$ is proportional to $\mathrm{Sylv}^{j,0}(P,Q)(U)$, $j=k+\ell$ in all cases
- We proved that $\mathrm{Sylv}^{j,0}(P,Q)(U)$ and $Sres_j(P,Q)(U)$ are proportional for $q \le j \le p$ and satisfy a similar induction for $j < q$ when we replace P, Q by Q, R
- We used this result to prove by induction that $\mathrm{Sylv}^{j,0}(P,Q)(U)$ and $Sres_j(P,Q)(U)$ are proportional !
- Introducing Multsymmetric Hermite Interpolation and Multi Sylvester double sums (generalizing the use of Multisymmetric Lagrange Interpolation [KSV]) and using the Exchange Lemma from [KSV] was essential in our proofs.

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