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Classification of $\aleph_0$*-categorical* C*-minimal pure* C*-sets*,
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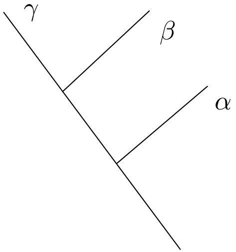
C-set

A $\underline{C\text{-relation}}$ is a ternary relation satisfying the four axioms:
1. $C(x,y,z) \to C(x,z,y)$
2. $C(x,y,z) \to \neg C(y,x,z)$
3. $C(x,y,z) \to [C(x,y,w) \vee C(w,y,z)]$
4. $x \neq y \to C(x,y,y)$.

A C-set is a set equipped with a C-relation.

Examples:

- Trivial C-relation: $C(x,y,z)$ si $x \neq y = z$.
- C-relation from an order: $C(x,y,z)$ if $(x < y$ and $x < z)$ or $y = z \neq x)$
- C-relation from an ultrametric distance $(d(x,y) \leq max(d(x,z), d(y,z)))$: $C(x,y,z)$ if $d(x,y) < d(y,z)$
- Tree



$C(\alpha,\beta,\gamma)$ if α and β "branch" below β and γ

Good trees

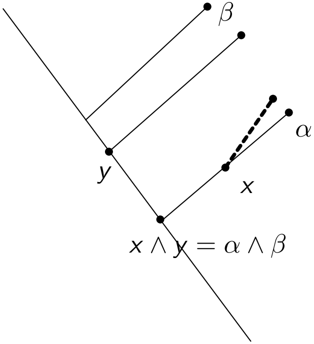
A good tree is an ordered set satisfying:

- For all $x \in T$, the set $\{y; y \leq x\}$ is linearly ordered
- it is a meet semi-lattice i.e. any two elements $x \neq y$ have an infimum, or meet, $x \wedge y$
- it has maximal elements, or leaves, everywhere (i.e. $\forall x, \exists y \ (y \geq x \wedge \neg \exists z > y))$
- any of its elements is a leaf or a node (i.e. of form $x \wedge y$ for some distinct x and y).

$$(T, \leq, \wedge, L, N)$$

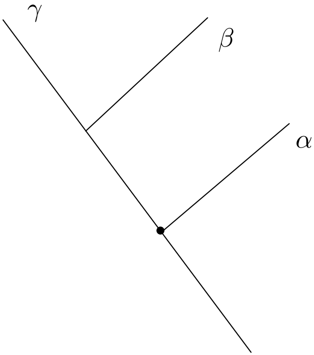
N : set of nodes

L: set of leaves



C-set $M(T)$

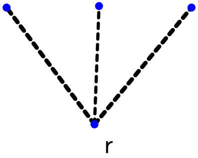
Let T be a good tree.
Let α be a leaf and let br$(\alpha) =: \{x \in N; x < \alpha\} \cup \{\alpha\}$ be the branch of α.
The set $M(T)$ of the branches with a leaf is identified with the set L of leaves and carries a canonical C-relation:



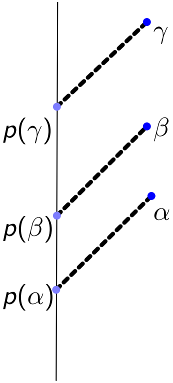
$C(\alpha, \beta, \gamma)$ if $\alpha \cap \beta = \alpha \cap \gamma \subset \beta \cap \gamma$

Good tree $T(M)$

Conversely, given a C-set M, there is a unique good tree $T(M)$ such that M is isomorphic to the set of branches with a leaf equipped with the canonical C-relation (Adeleke-Neumann, 98).
The set of leaves of $T(M)$ is M
The set of nodes is a quotient of M^2.
Examples:



trivial relation



linear order

In these examples all the leaves are isolated i.e. have a predecessor.

C-sets and good trees are bi-interpretable classes.

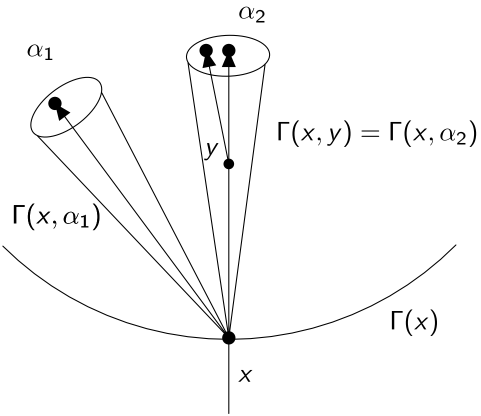
Let M be a C-set then, M and $L(T(M))$ (the set of leaves of $T(M)$) are definably isomorphic.
Let T be a good tree then T and $T(M(T))$ are definably isomorphic.

C-minimality

(Haskell, Macpherson, Steinhorn,94)
A C-$\mathrm{structure}$ is a C-set possibly equipped with additional structure.
A C-structure $\mathcal{M}$ is called C-$\mathrm{minimal}$ if any definable subset of $\mathcal{M}$ is definable by a quantifier free formula in the pure language $\{C\}$ and it holds for any elementary equivalent structure.

Cones and thick cones

Let M be a C-set, $T(M)$ its canonical good tree.
Notations: latin letters: x, y, ... for nodes in $T(M)$
greek letters: α, β, ... for leaves in $T(M)$ or elements of M.



In $T(M)$

- Cone of α at x: $\Gamma(x, \alpha) = \{t \in T(M); x \wedge t > x\}$ (= cone of y at x: $\Gamma(x, y)$
- Thick cone at x: $\Gamma(x) = \{t \in T(M); t \geq x\}$

In M:

- Cone of α at x: $\mathcal{C}(x, \alpha) = \{\beta \in M; \alpha \wedge \beta > x\}$
-Thick cone at x: $\mathcal{C}(x) = \{\beta \in M; \beta \wedge x = x\}$

Cones and thick cones

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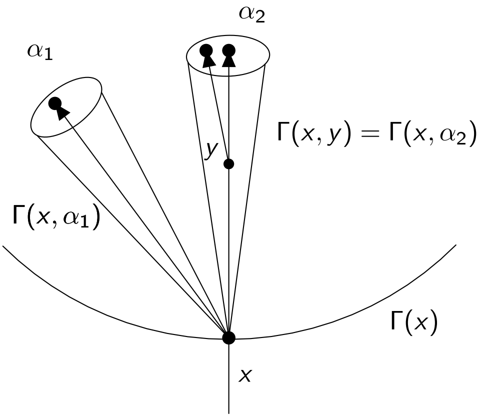

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Characterization of C-minimality

A C-structure is C-minimal iff any definable subset of M is a boolean combination of cones and thick cones.

Haskell-Macpherson, 94: If a C-structure M is C-minimal, each branch $br(\alpha)$ with a leaf of $T(M)$ is o-minimal in $T(M)$.

Purpose of the talk

We say that a structure is $\aleph_0$-categorical if its complete theory has only one countable model (up to isomorphism).

We say that a structure is *indiscernible* if it realizes only one complete 1-type over $\emptyset$.
Let M be a $\aleph_0$-categorical C-minimal structure then by Ryll-Nardzewski's Theorem: M is a finite union of indicernible subsets.

Article outline:

I) Classification of $\aleph_0$-categorical C-minimal and **indiscernible** pure C-sets

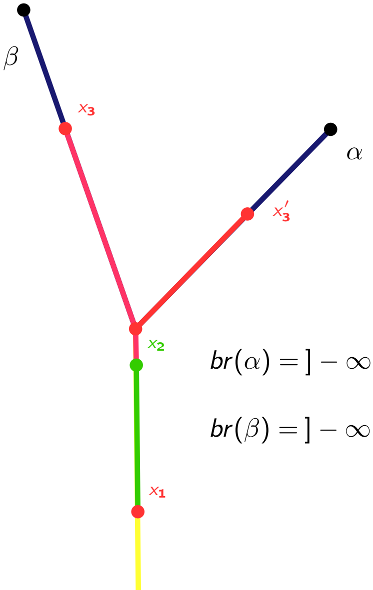
II) General classification

I am only going to talk about part I.
So, from now on $\mathcal{M}$ is an indiscernible, finite or $\aleph_0$-categorical pure C-structure, and $T(M)$ is its canonical good tree.

Uniform decomposition of the branches $T(M)$ into one-typed intervals

Theorem 1

Let $\mathcal{M}$ be an indiscernible finite or $\aleph_0$-categorical C-structure, Let $T(M)$ be its canonical good tree. Then there exists an integer $n \geq 1$ such that for any leaf α of T, $br(\alpha) = \bigcup_{j=1}^{n} I_j(\alpha) \cup \{\alpha\}$ with $I_j(\alpha) < I_{j+1}(\alpha)$, where the I_j are one-typed intervals. This decomposition is unique if we assume that the $I_j(\alpha)$ are maximal one-typed, Possible forms of each $I_j(\alpha)$ are $\{x\}$, $]x, y[$ and $]x, y]$. The decomposition is independent of the leaf α,and coincide below the meeting node of two leaves.



$$br(\alpha) =]-\infty, x_1[\cup \{x_1\} \cup]x_1, x_2] \cup]x_2, x_3] \cup]x_3, \alpha[\cup \{\alpha\}$$

$$br(\beta) =]-\infty, x_1[\cup \{x_1\} \cup]x_1, x_2] \cup]x_2, x_3'] \cup]x_3', \beta[\cup \{\beta\}$$

Proof ingredients

- Ryll-Nardzewski Theorem and $\aleph_0$-categoricity imply a finite number of p-types over $\emptyset$ in M.

- finite number of p-types in $T(M)$

- $T(M)$ is finite or $\aleph_0$-categorical.

- $N_T = N_1 \cup N_2 \cup \cdots \cup N_m$ where the nodes of N_i have the same complete type over $\emptyset$.

- For each α, $br(\alpha) \cap N_i$'s are definable.

- By o-minimality of the branches, for each α and each i, $br(\alpha) \cap N_i$ is a finite union of singletons and intervals.

- If such an interval has a first element then this interval is in fact a singleton.

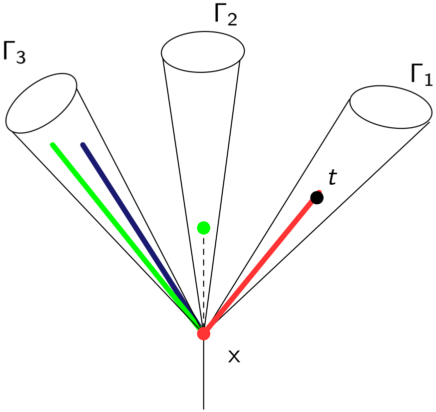
Hence, for a given leaf α, $br(\alpha)$ is the order sum of finitely many maximal one-typed intervals, and by indiscernibility, the number of such one-typed intervals, the form of each of them, and the tree-type of its elements, depend only on its index and not on the branch.

Inner and border cones

Inner cone: We say that a cone Γ at x is an $\mathrm{inner\ cone}$ if the two following conditions are realized:

1. x has no successor on any branch of Γ.
2. There exists $t \in \Gamma$ (equivalently for all t) s.t., for any $t' \in T$ with $x < t' < t$, t' is of same tree-type as x.

Border cone: Otherwise, we say that Γ is a $\mathrm{border\ cone}$.



Γ_1 is an inner cone, Γ_2 and Γ_3 are border cones.

Color of a node - One-colored intervals

The color of a node x of a tree T is the ordered pair $(m, \mu) \in (\mathbb{N} \cup \{\infty\})^2$ where m is the number of border cones at x and μ the number of inner cones at x.
The color of a node of $T(M)$ is $\emptyset$-definable in the pure order of $T(M)$

same type $\Rightarrow$ same color

One-colored interval on a branch $br(\alpha)$:
- (0): $I = \{x\}$ where x if of color $(m, 0)$, $m \geq 2$
- (1.a): $I =]x, y[$ where any element of I is of color $(0, \mu)$, $\mu \geq 2$
-(1.b): $I =]x, y]$ where any element of I is of color (m, μ), $m, \mu \geq 1$.

a one-typed interval is a one-colored interval

Precolored good trees

In order to describe the theory of the canonical tree of an indiscernible $\aleph_0$-categorical or finite C-minimal C-structure, we define now precolored good trees which are constructed from the conclusion of Theorem 1, replacing "one-typed interval" by the (in general different) notion of "one-colored interval".

We say that T is a precolored good tree if there is no node of color (∞, ∞) and there exists an integer n, such that for all $\alpha \in L$:

(1) the branch $br(\alpha)$ can be written as a disjoint union of its leaf and n one-colored intervals $br(\alpha) = \cup_{j=1}^{n} I_j(\alpha) \cup \{\alpha\}$, with $I_j(\alpha) < I_{j+1}(\alpha)$.

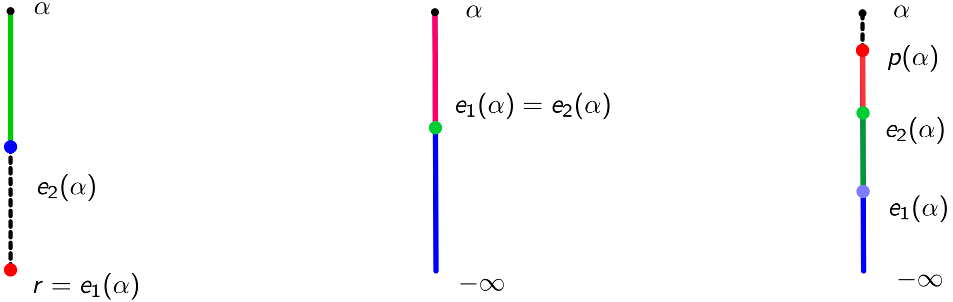
(2) The $I_j(\alpha)$ are maximal one-colored, that is, $I_j(\alpha) \cup I_{j+1}(\alpha)$ is not a one-colored interval, and for all $j \in \{1, \cdots, n\}$, the color of $I_j(\alpha)$ is independent of α.

(3) For any $\alpha, \beta \in L$ and $j \in \{1, \cdots, n\}$, if $\alpha \wedge \beta \in I_j(\alpha)$, then $\alpha \wedge \beta \in I_j(\beta)$, $I_j(\alpha) \cap I_j(\beta)$ is an initial segment of both $I_j(\alpha)$ and $I_j(\beta)$; and for any $i < j$, $I_i(\alpha) = I_i(\beta)$.

The integer n, which is unique by maximality of the basic one-colored intervals, is called the depth of the precolored good tree T.

Examples

3 examples of a branch in the case of a precolored of depth 3

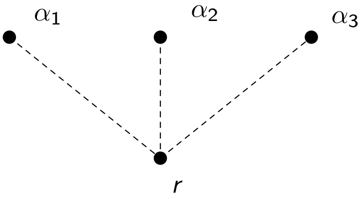


We denote by $e_j(\alpha)$ the lowest bound of the interval I_j and by $p(\alpha)$ the predecessor of α if it is isolated.

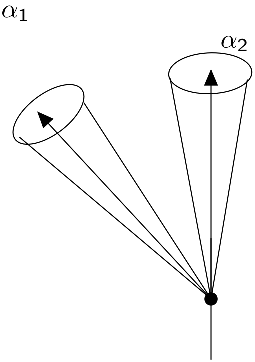
By Theorem 1 if M is C-minimal, $\aleph_0$-categorical or finite and indiscernible then $T(M)$ is precolored.

Precolored trees of depth 1

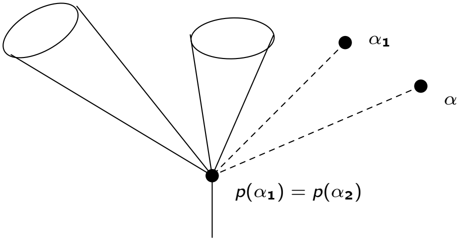
A border cone is an isolated leaf
An inner cone is infinite cone



Form (0) $m = 3, \mu = 0$



Form (1.*a*) $m = 0, \mu = 2$



Form (1.*b*) $m = 2, \mu = 2$

1-colored trees

Définition 1

T is 1-colored when T is of the form:

(0) a root and m isolated leaves ($m \geq 2$)

(1.a) for any leaf α of T, $]-\infty, \alpha[$ is densely ordered and at each node of T there are exactly $\mu \geq 2$ cones, all infinite.

(1.b) for any leaf α of T, α has a predecessor, the node $p(\alpha)$, $]-\infty, p(\alpha)]$ is densely ordered and at each node of T there are exactly m isolated leaves and μ infinite cones.

The ordered pair (m, μ) is called the branching color of the 1-colored tree T.

In the cases (0) and $(1.b)$ all leaves are isolated, in the cas $1.a)$ all leaves are non isolated.
We call the ordered pair (m, μ) the branching color.

A precolored tree of depth 1 is a one colored good tree.

1-colored trees complete theories

For m and μ in $\mathbb{N} \cup \{\infty\}$ such that $m + \mu \geq 2$, we denote by $\Sigma_{(m,\mu)}$ the set of axioms in the language $\mathcal{L}_1 := \{L, N, \leq, \wedge\}$ describing 1-colored good trees of branching color (m, μ), and by S_1 the set of all these $\mathcal{L}_1$-theories, $S_1 := \{\Sigma_{(m,\mu)} : (m, \mu) \in (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$ with $m + \mu \geq 2\}$.

Theorem 2

Any theory in S_1 is $\aleph_0$-categorical, hence complete. Moreover, it admits quantifier elimination in a natural language, $\Sigma_{(m,0)}$ in $\{L, N\}$, $\Sigma_{(0,\mu)}$ in $\mathcal{L}_1$ and $\Sigma_{(m,\mu)}$ with $m, \mu \neq 0$ in $\mathcal{L}_1^+ = \{L, N, \leq, \wedge, p\}$

$$\begin{tikzcd}
M \textit{ is } \text{C-min., } \aleph_0\text{-categ. indiscernible with all the nodes of same type} \arrow[r, Rightarrow] & T(M) \text{ is precolored of depth 1} \arrow[d, Rightarrow] \\
& T(M) \text{ is 1-colored} \arrow[ul, Rightarrow]
\end{tikzcd}$$

For any node of a 1-colored good tree:
Color = branching color = type

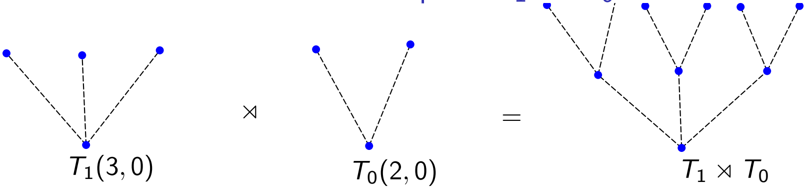
Extension $T \rtimes T_0$

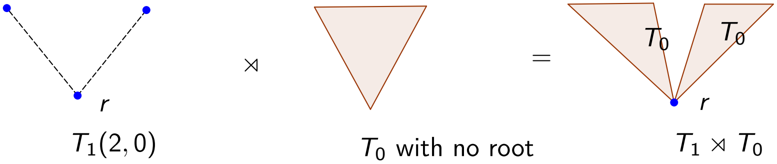
Let T and T_0 be two trees. We define roughly $T \rtimes T_0$, the "extension of T by T_0", as the tree consisting of T in which each leaf is replaced by a copy of T_0.
We require some conditions:

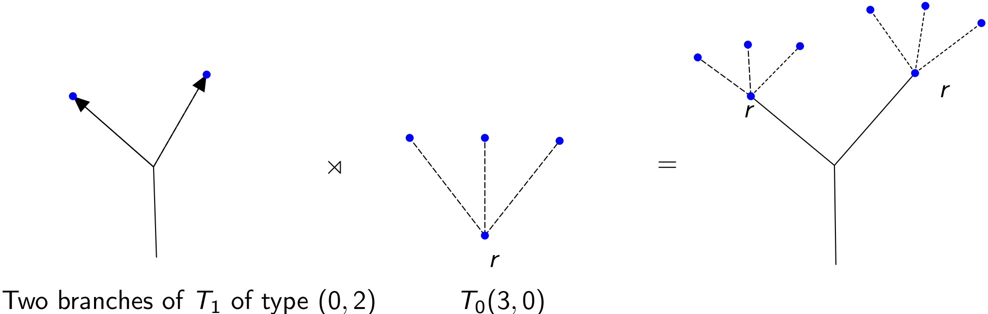
- T_0 is a 1-colored good tree
- Either all leaves of T are isolated or all leaves of T are non isolated.
- If T has non isolated leaves, T_0 should have a root.

Let L_T and N_T be respectively the set of leaves and nodes of T, L_0 and N_0 the set of leaves and nodes of T_0.
As a set, $T \rtimes T_0$ is the disjoint union of N_T and $L_T \times T_0$.
$N_{T \rtimes T_\mathbf{0}} = N_T \cup \{(\alpha, t), \alpha \in L_T,\ t \in N_0\}$
$L_{T \rtimes T_\mathbf{0}} = \{(\alpha, \beta), \alpha \in L_T,\ \beta \in L_0\}$

Examples $T_1 \rtimes T_0$

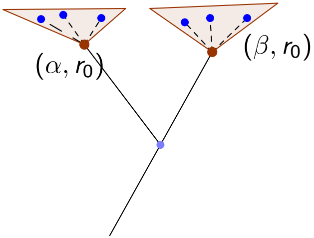




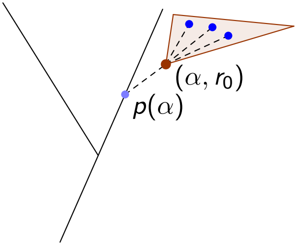


Equivalence relation on $T \rtimes T_0$

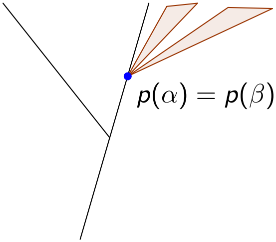
The set of nodes N_T embedds canonically in $T \rtimes T_0$, and for each leave α there is an embedding of T_0 "above" α.
However, in the case where T_0 has no root, T does not appear as a subset of $T \rtimes T_0$ but as a quotient of $T \rtimes T_0$.
More precisely, if we define an equivalence relation on $T \rtimes T_0$ by for any $x \in N_T$, $cl(x) = \{x\}$ and for any $(\alpha, t) \in L_T \times T_0$, $cl(\alpha, t) = \{\alpha\} \times T_0$, then $T \rtimes T_0 / \sim$ and T are isomorphic good trees.



T with non isolated leaves
T_0 with a root



T with isolated leaves
T_0 with a root



T with isolated leaves
T_0 with no root

Langage and theory of $T \rtimes T_0$

For simplicity, I suppose now that T and T_0 are 1-colored good trees with complete theory Σ and Σ_0 respectively.
Let $\mathcal{L}_2^+ = \{\leq, \wedge, N, L, e, E, E_{\geq}, p\}$ where e is a partial function, $E = Im(e)$, $E_{\geq} = Dom(e)$. For each $\alpha \in L_T$, $e(\alpha)$ must be interpretated as the node where T_0 plugs on $br(\alpha)$.
More precisely:
If T_0 has a root r_0, $Dom(e) = L_T \times T_0$, $e(\alpha, t) = (\alpha, r_0)$.
If T_0 has no root (recall that in this case all the leaves of T must be isolated),
$Dom(e) = L_T \times T_0 \cup p(L_T)$, $e(\alpha, t) = p(\alpha)$

Theory $\Sigma \rtimes \Sigma_0$. :
- Good trees
- $E_{\geq} = Dom(e)$
- $E = Im(e)$
-$L \subset Dom(e)$, $E \cap L = \emptyset$
-$\forall x \in Dom(e), e(x) \leq x$
- for all x, $cl(x) = \{x\}$ ou $cl(x) \equiv T_0$
- $T \rtimes T_0/\sim \equiv T$

$\Sigma \rtimes \Sigma_0$ is a complete axiomatization of the theory of $T \rtimes T_0$,
has a unique finite or countable model,
eliminates quantifier in $\mathcal{L}_2^+$,
$M(T \rtimes T_0)$ est C-minimal.

Induction: solvable trees

A solvable good tree is either a singleton or a tree of the form $(\dots(T_1 \rtimes T_2) \rtimes \cdots T_{n-1}) \rtimes T_n$ for some integer $n \geq 1$, where $T_1, \cdots, T_n$ are 1-colored good trees such that, for each i, $1 \leq i \leq n-1$, if T_i is of type $(1.a)$ then T_{i+1} is of type (0).

- Associativity: we write $T_1 \rtimes T_2 \rtimes \cdots \rtimes T_n$.
- Difficulty: a solvable good tree may have decompositions into iterated extensions of 1-colored good trees of different lengths. So we have to deal with some exceptions.

Let T be a solvable good tree, then there exists a unique $n \in \mathbb{N}$ such that T is an n-solvable good tree, i.e of the form $T_1 \rtimes T_2 \rtimes \cdots \rtimes T_n$ with some additional conditions. We call T a n-solvable good tree.

Langage and theory of *n*-solvable good trees

Langage: $\mathcal{L}_n^+ = \{\leq, \wedge, L, N, (e_i)_{i=1}^{n-1}, (E_i)_{i=1}^{n-1}, (E_{\geq,i})_{i=1}^{n-1}, p\}$ where the partial functions e_i's are interpreted as the nodes where the trees T_{i+1} "branche" on the tree $T_1 \rtimes \cdots \rtimes T_i$, in other words the images of the functions e_i indicate the changing of color.

Let T be an n-solvable good tree. Then
- T eliminates quantifiers in the language $\mathcal{C}_n^+$,
- functions and predicates of $\mathcal{L}_n$ are definable in the pure order,
- T is finite or $\aleph_0$-categorical,
- $M(T)$ is indiscernible and C-minimal.

Let $T_1, T_2, \cdots, T_n$ be 1-colored good trees (neither realizing exceptions). Let $\Sigma_1, \Sigma_2, \cdots, \Sigma_n$ be their theories in the language $\mathcal{L}_1$ and $\Sigma_1 \rtimes \cdots \rtimes \Sigma_n$ the $\mathcal{L}_n$-theory defined by induction.

We denote by S_n, $n \geq 1$, the set of all theories $\Sigma_1 \rtimes \Sigma_2 \rtimes \cdots \rtimes \Sigma_n$ in the language $\mathcal{L}_n^+$ and by S_0 the $\mathcal{L}_1$- theory of the singleton.

Colored good trees

For $n \in \mathbb{N} \cup \{\infty\}$, we call n-colored any model of a given theory of S_n.

For any integer n any theory in S_n is complete and admits quantifier elimination in the language $\mathcal{L}_n$. Furthermore S_n is the set of all complete theories of n-colored good trees.

Classifcation of indiscernible, $\aleph_0$-categorical, C-minimal pure C-sets

Theorem 3

Let M be a pure C-set. Then the following assertions are equivalent:

(i) *M is finite or $\aleph_0$-categorical, C-minimal and indiscernible*

(ii) *$T(M)$ is a precolored good tree.*

(iii) *$T(M)$ is a colored good tree.*