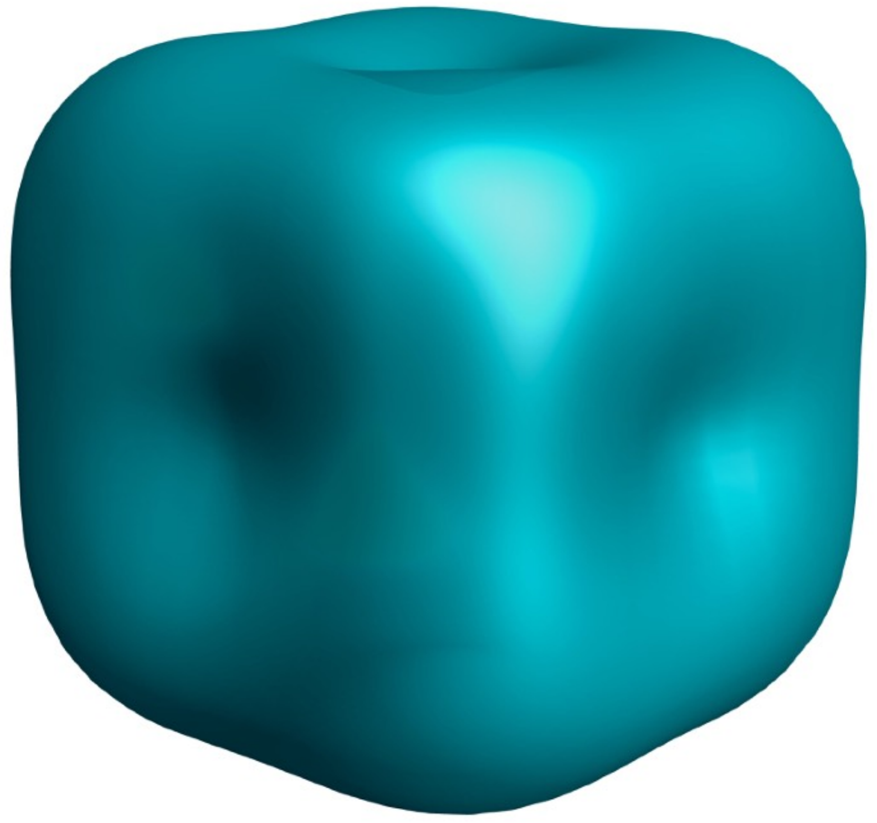
Effective polynomial approximation of starshaped sets



Chiara Meroni

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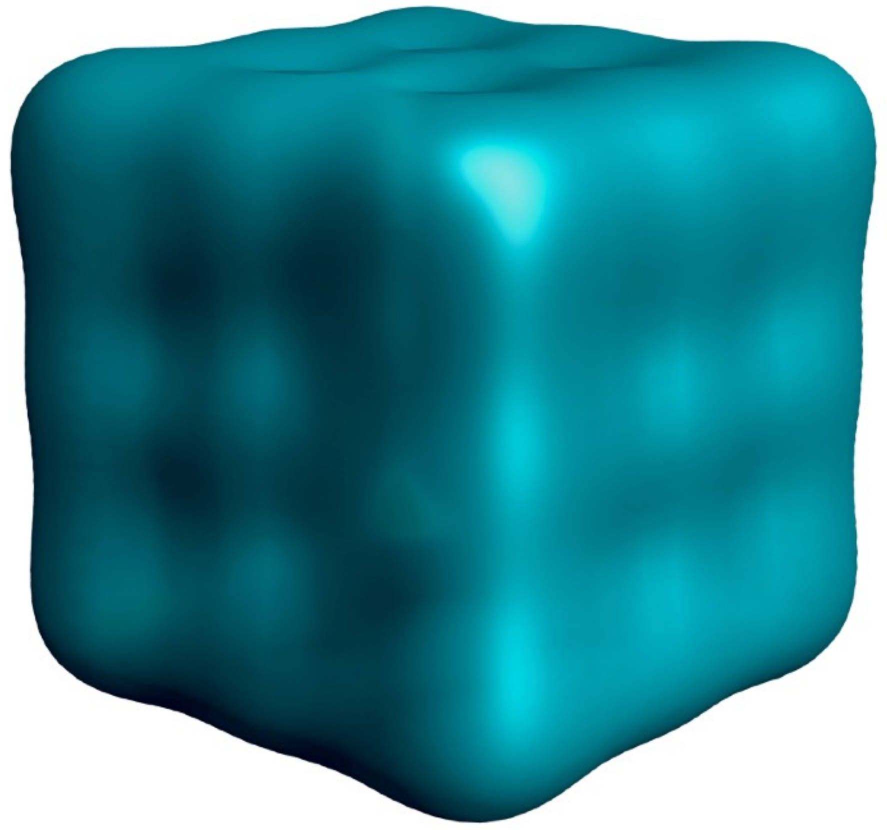

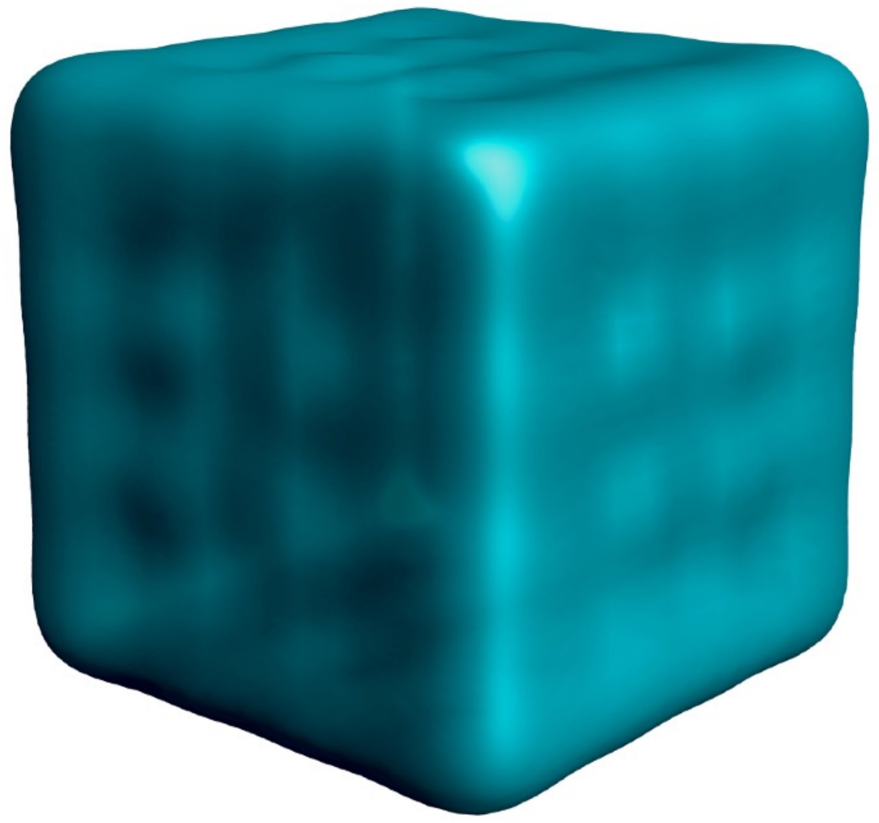

joint work with
Jared Miller and Mauricio Velasco

DDG40: Structures Algébriques et Ordonnées

Busemann–Petty problem

1956 ⟶ 1999

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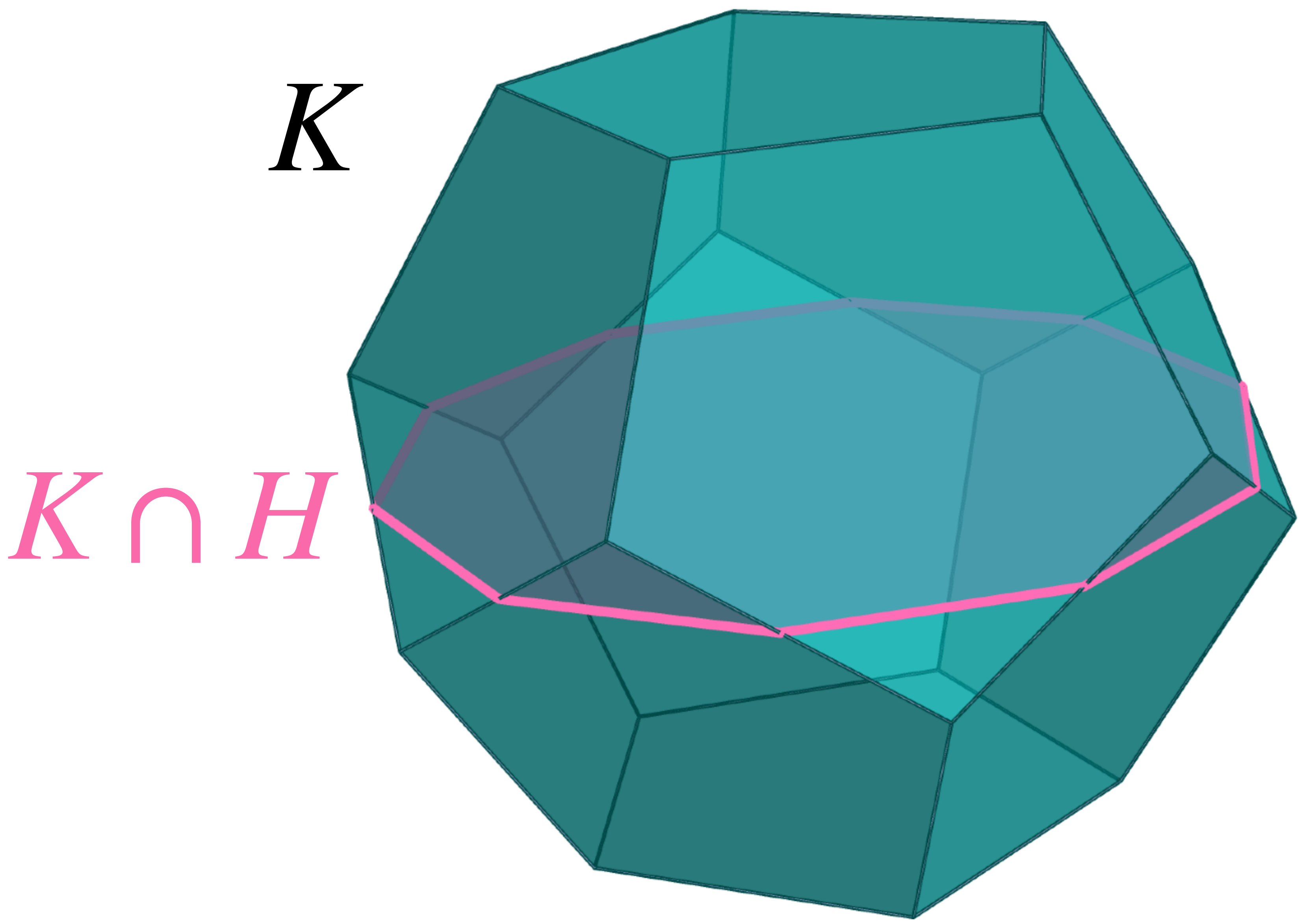
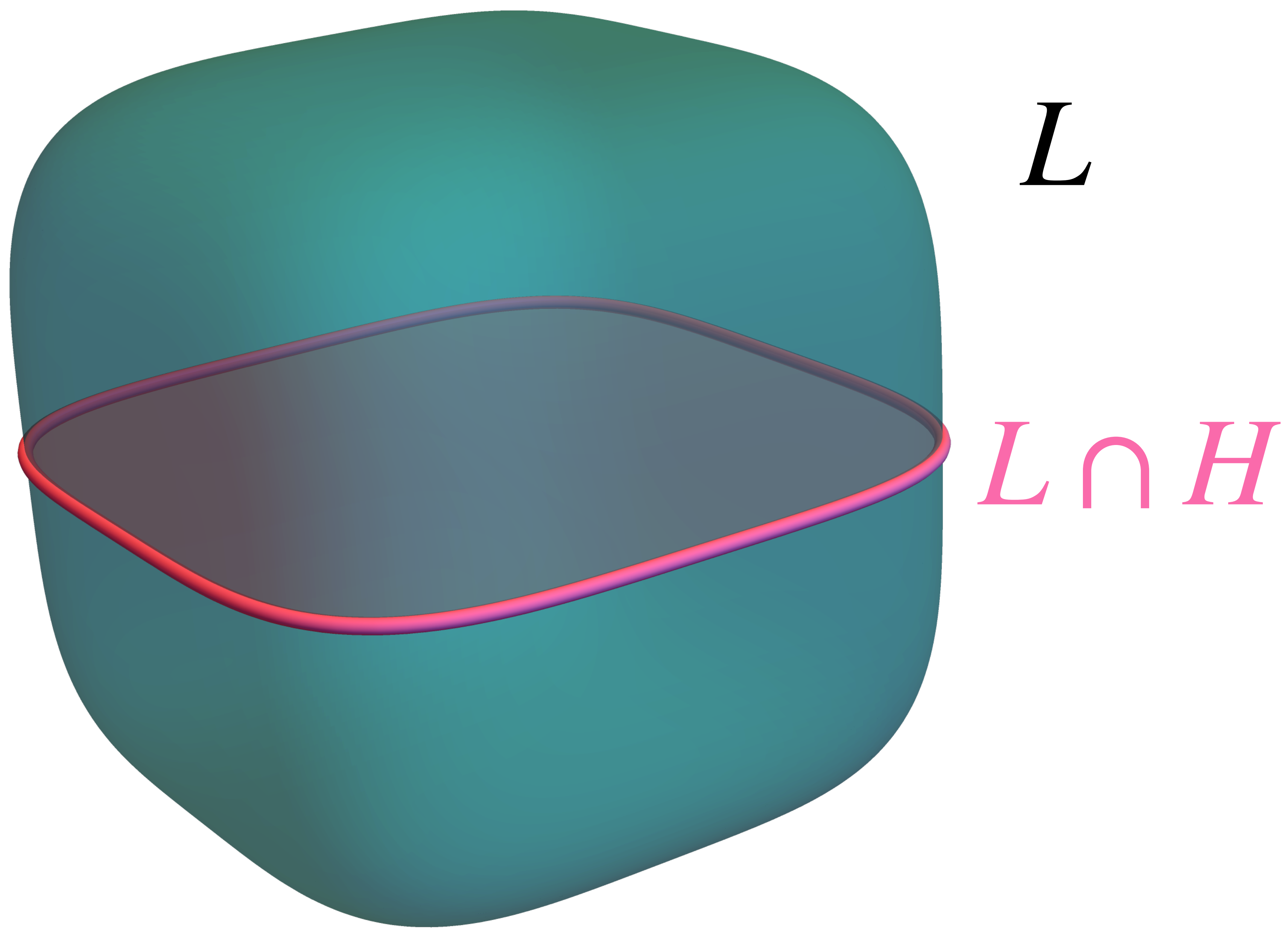
Let K and L be convex bodies in $\mathbb{R}^n$. Assume that for every hyperplane H through the origin,

$$\operatorname{vol}(K \cap H) \leq \operatorname{vol}(L \cap H).$$





Does this imply that $\operatorname{vol}(K) \leq \operatorname{vol}(L)$?

Busemann–Petty problem

$\mathrm{vol}\,(K) \leq \mathrm{vol}\,(L)$?

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In general NOT true!

It holds for convex bodies in $\mathbb{R}^n$ when $n \leq 4$.

Let's refine it:

Bourgain's slicing conjecture (1986):

$$\mathrm{vol}\,(K) \leq C\ \mathrm{vol}\,(L),$$

where C does not depend on n.

Rephrase:

Bourgain's slicing conjecture (1986):

Let $K \subset \mathbb{R}^n$ be a convex body of volume 1.
Does there exist a hyperplane H satisfying

$$\mathrm{vol}(K \cap H) > \frac{1}{C}$$

where C does not depend on n?

arXiv > math > arXiv:2412.15044

Affirmative Resolution of Bourgain's Slicing Problem using Guan's Bound

Boaz Klartag, Joseph Lehec

Building upon

arXiv > math > arXiv:2412.09075

A note on Bourgain's slicing problem

Qingyang Guan

Large volume slices

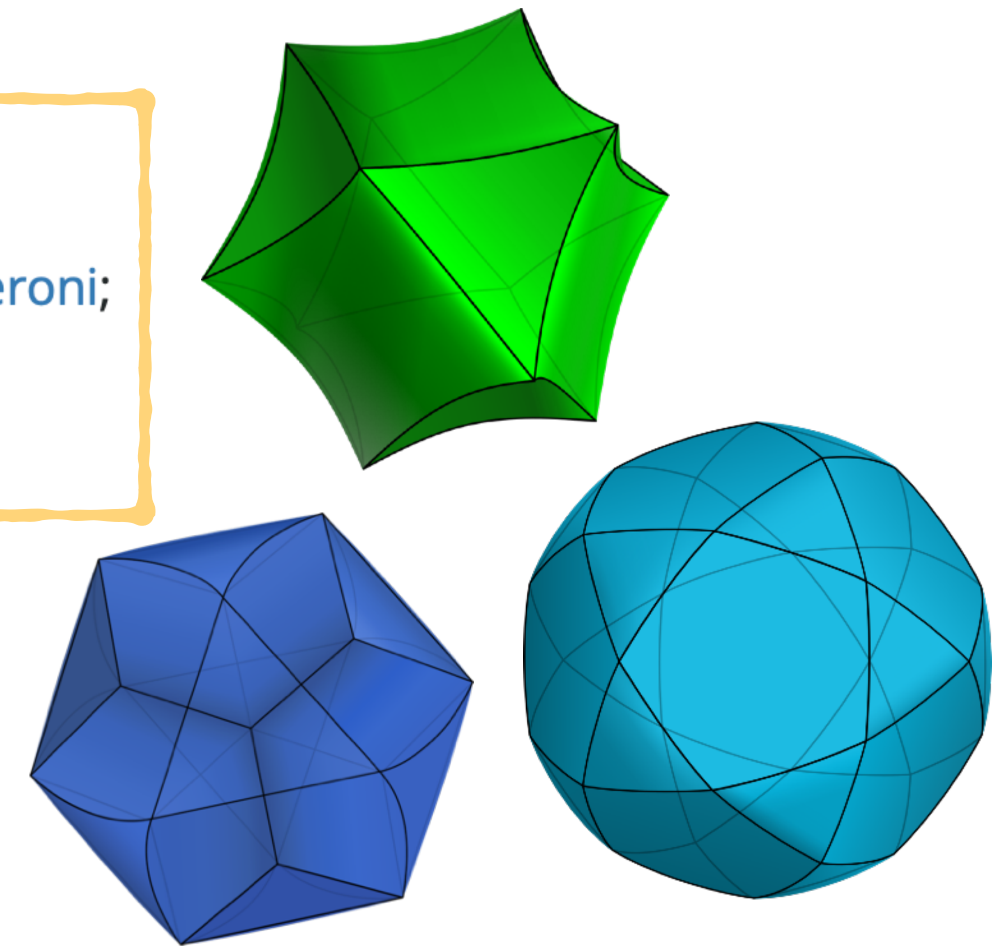
For polytopes: exact poly-time algorithms

The best ways to slice a Polytope

by Marie-Charlotte Brandenburg, Jesús A. De Loera and Chiara Meroni;

Math. Comp. **94** (2025), 1003-1042

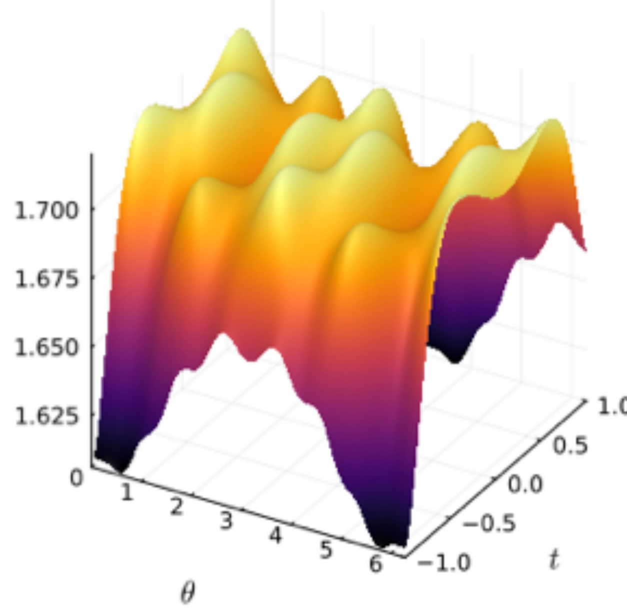
DOI: https://doi.org/10.1090/mcom/4006



arXiv > math > arXiv:2403.04438

Maximizing Slice-Volumes of Semialgebraic Sets using Sum-of-Squares Programming

Jared Miller, Chiara Meroni, Matteo Tacchi, Mauricio Velasco

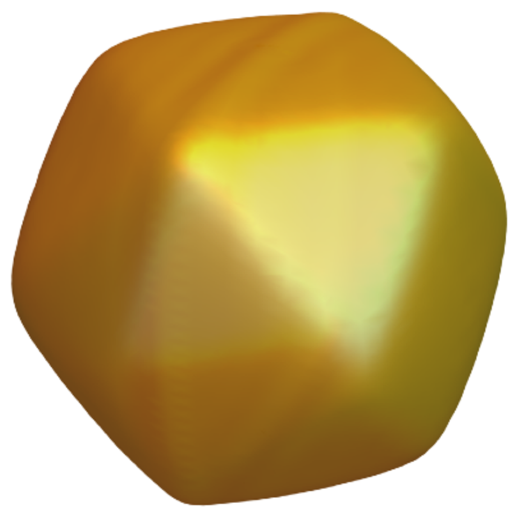


For convex bodies: approximation

arXiv > math > arXiv:2505.24352

Approximation of starshaped sets using polynomials

Chiara Meroni, Jared Miller, Mauricio Velasco



Starbodies

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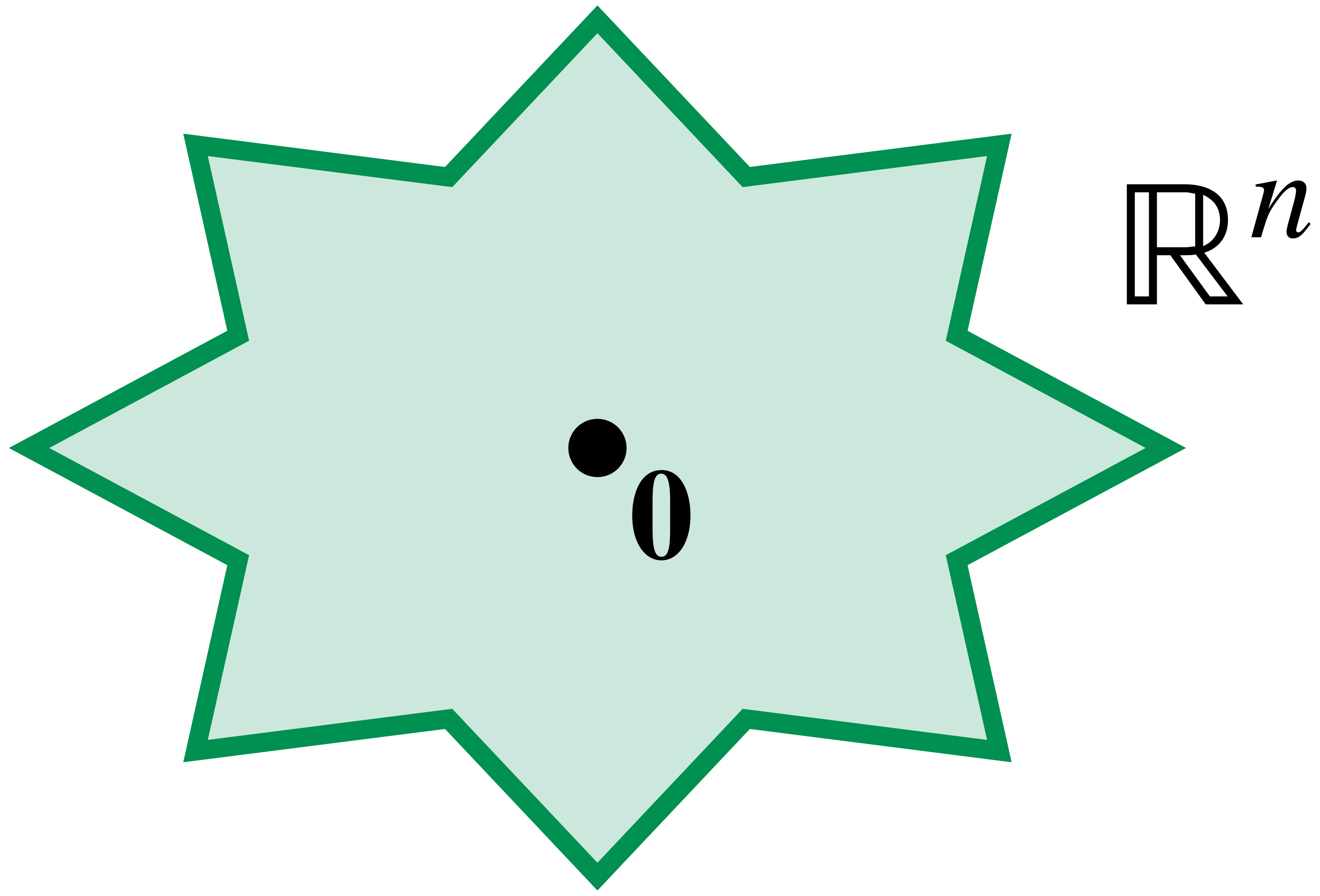
Starshaped set (w.r.t. the origin): $x \in L \Rightarrow [\mathbf{0}, x] \subset L$

Starbody (w.r.t. the origin): compact starshaped set

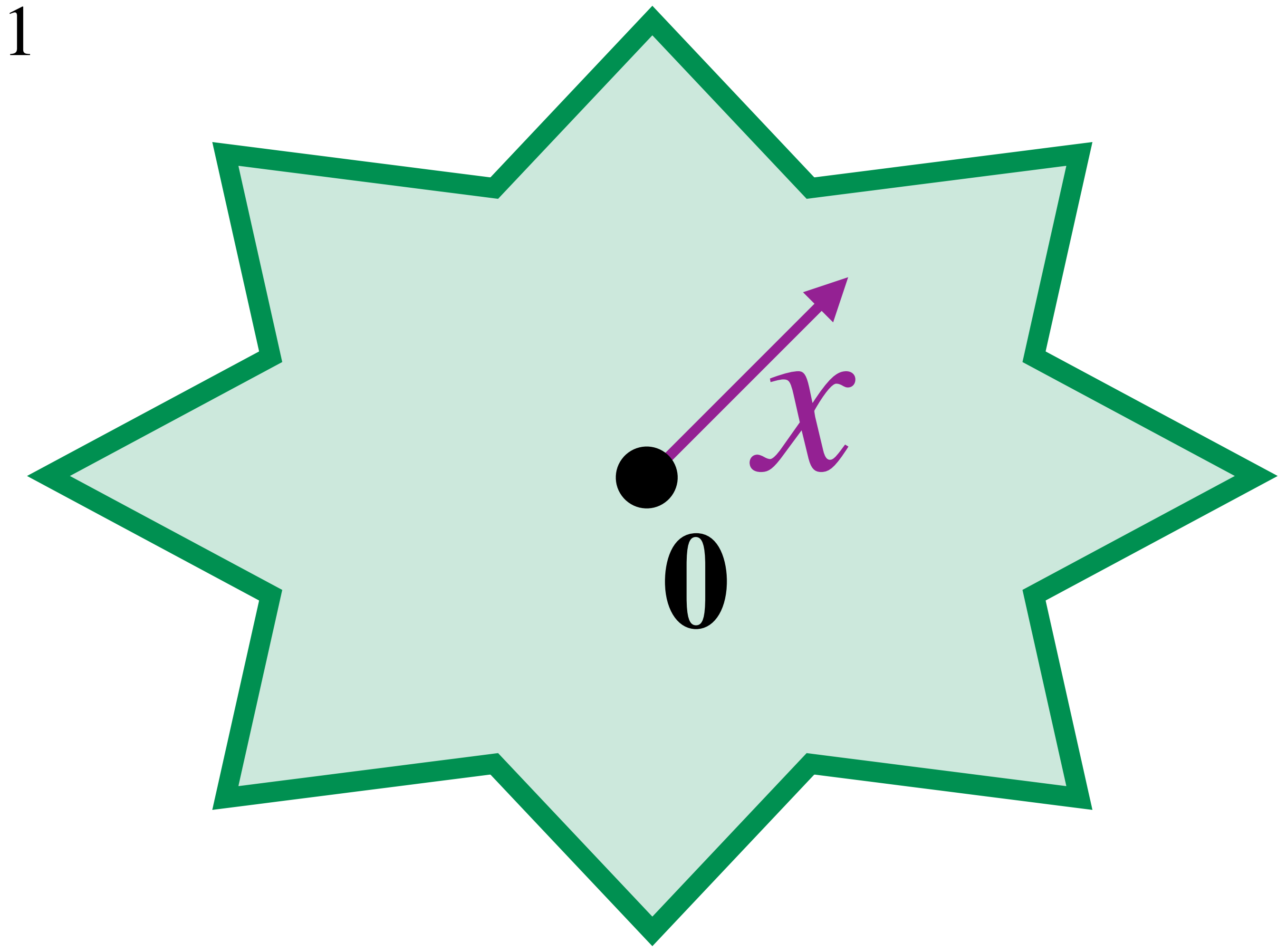


Q:
- How can we describe or approximate starbodies?
- How can we compute natural invariants efficiently?
- How can we model the space of starbodies?

Radial & gauge functions

Radial function: $\rho_L(x) = \max\{\lambda \in \mathbb{R}_{>0} \,|\, \lambda x \in L\}$, for all $x \in S^{n-1}$

Gauge function: $\gamma_L(x) = \dfrac{1}{\rho_L(x)}$, for all $x \in S^{n-1}$



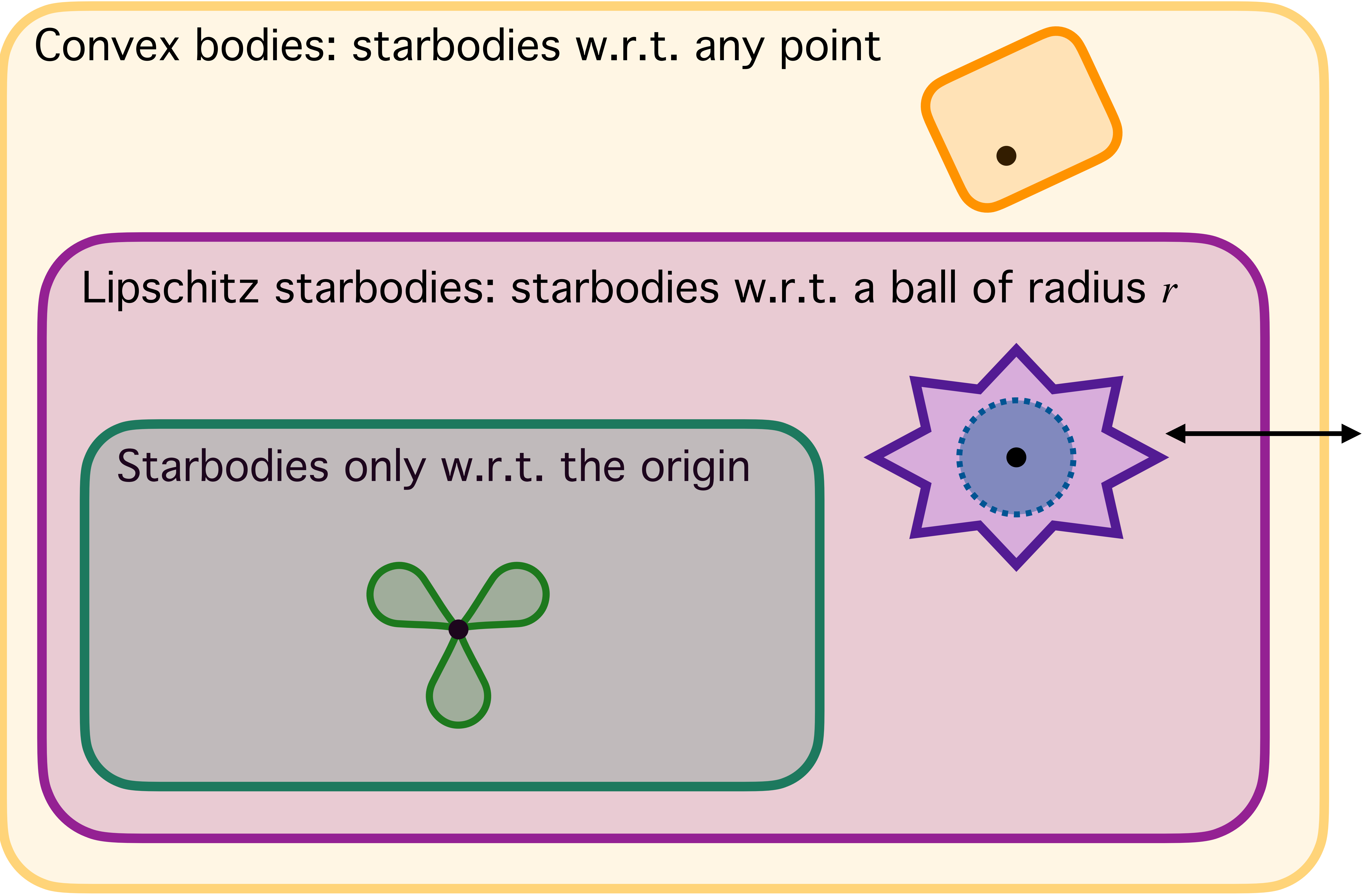
$\left\{ \text{Nonnegative functions on } S^{n-1} \right\} \longleftrightarrow \left\{ \text{starbodies w.r.t. the origin} \right\}$

can be wild!

For $P = \{x \in \mathbb{R}^n \,|\, Ax \leq 1\}$ we have

$$\gamma_P(x) = \max_i \; A_i \cdot x \;, \qquad \rho_P(x) = \min_i \left\{ \frac{1}{A_i \cdot x} \;\middle|\; A_i \cdot x > 0 \right\}$$

Nice starbodies



Boulkhemair, Chakib, Sadik (2023)

$\left\{\begin{array}{c}\text{Positive, Lipschitz continuous} \\ \text{Gauge function on } S^{n-1} \text{ with} \\ \text{valid Lipschitz constant } 1/r\end{array}\right\}$

Even nicer starbodies

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A polystar body is a starbody whose radial or gauge function is the restriction of a multivariate polynomial to S^{n-1}

$$\left\{ \text{polyradial starbodies} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{nonnegative} \\ \text{polynomials on } S^{n-1} \end{matrix} \right\} \longleftrightarrow \left\{ \text{polygauge starbodies} \right\}$$

whose radial function is polynomial

whose gauge function is polynomial

Polystar bodies

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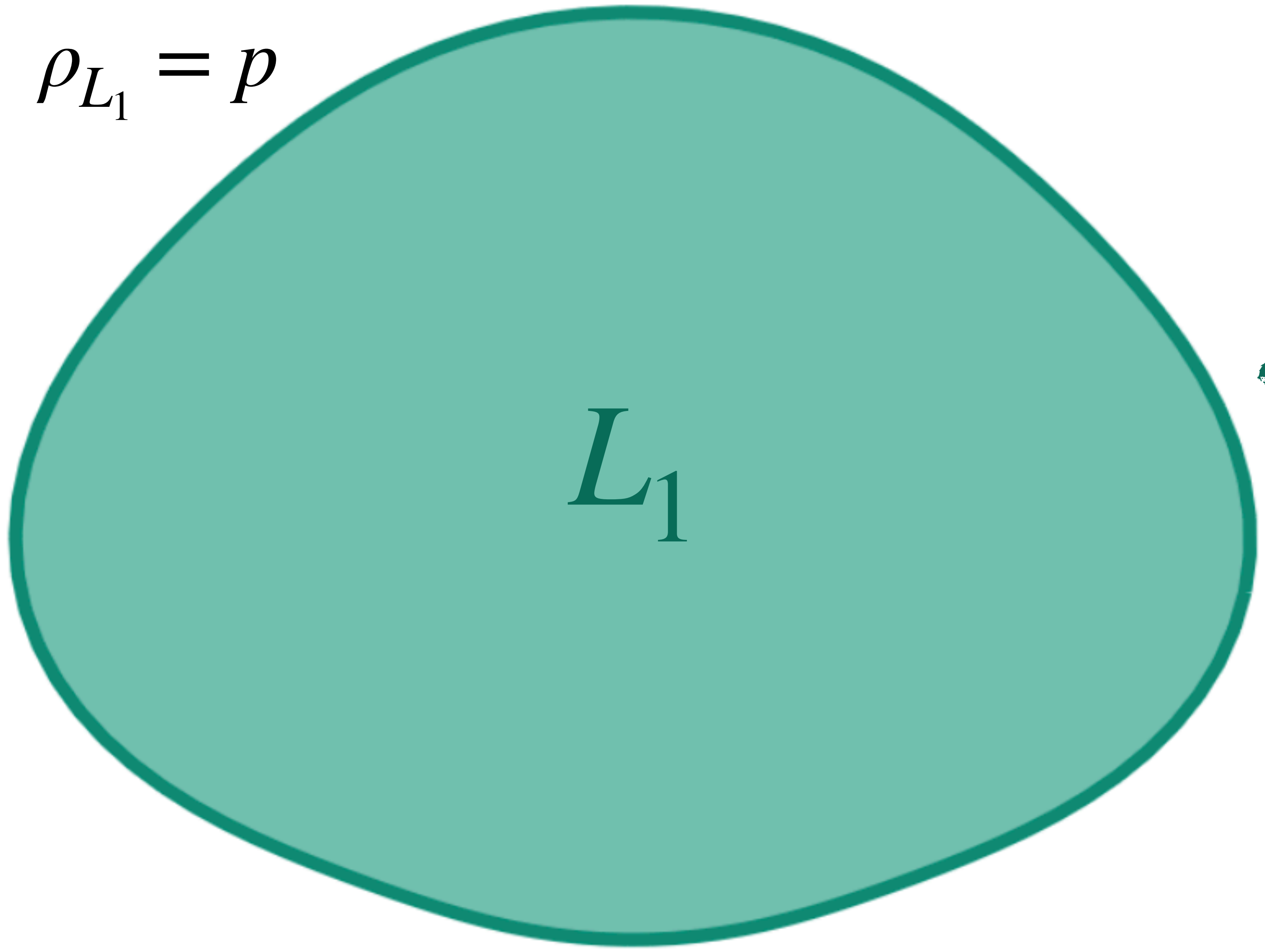
A polystar body is a starbody whose radial or gauge function is the restriction of a multivariate polynomial to S^{n-1}

$$p = 32x^6 + 32y + 128 = 32((\cos\theta)^6 + \sin\theta + 4)$$

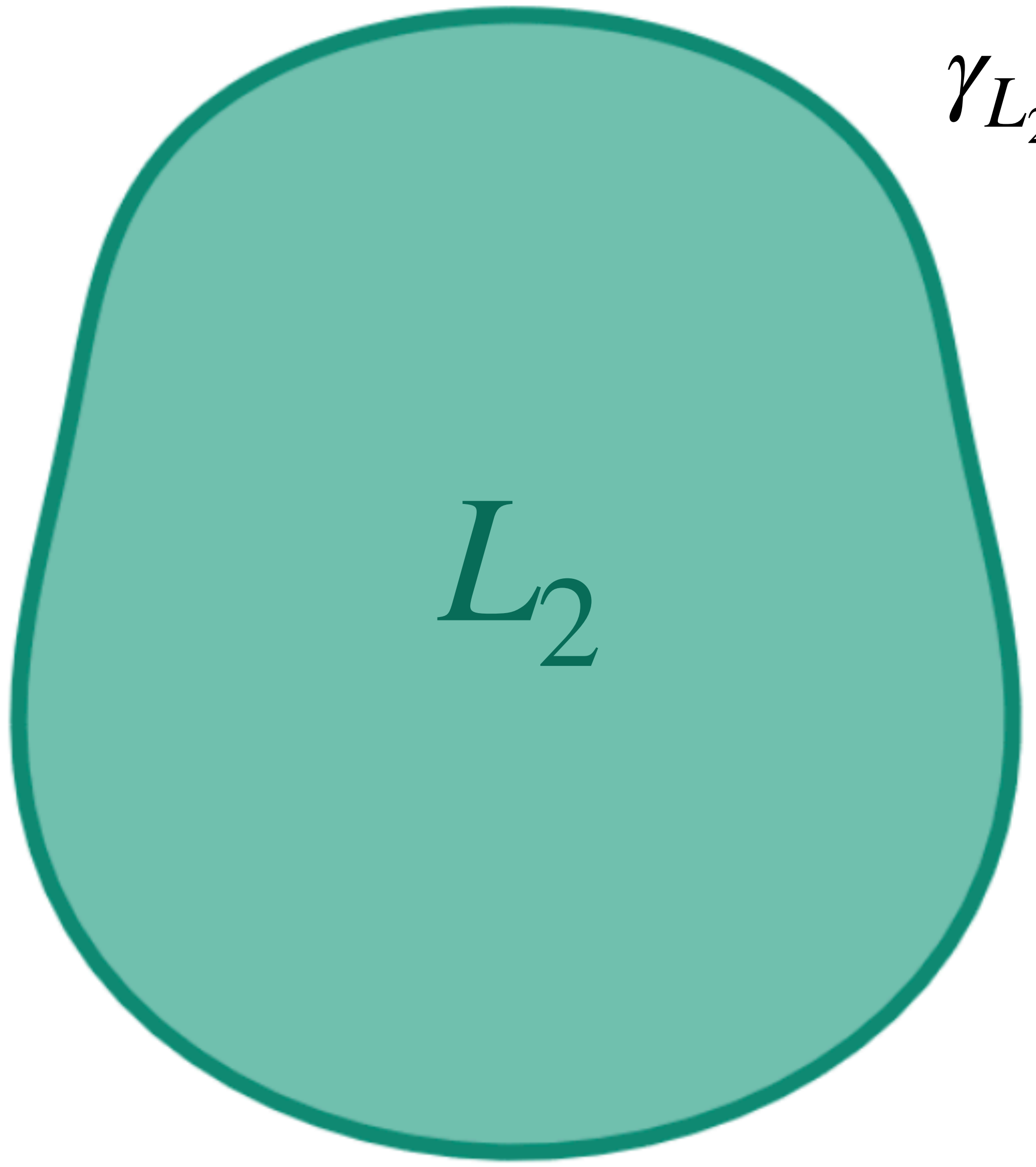
Polyradial

$\rho_{L_1} = p$



Polygauge

$\gamma_{L_2} = p$



Density

Theorem [M,M,V]: The set of polyradial/polygauge bodies is dense in the set of starbodies with continuous radial/gauge function.

Sure, Stone-Weierstrass Theorem

Multiple levels:

* Theoretical ✓
* Theoretical constructive → How can we approximate a starbody using polystar bodies?
* Sharp approximation guarantees
* Computational

Approximation

$$\|f-g\|_\infty = \sup_{x\in S^{n-1}} |f(x)-g(x)|$$

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Theorem [M,M,V]: Let f be a Lipschitz function with Lipschitz constant κ on S^{n-1}. Then, there exists an explicit sequence of univariate nonnegative polynomials $\{u_d\}_d$ with $u_d : [-1,1] \to \mathbb{R}$ of degree d such that

$$\|f - T_{u_d}(f)\|_\infty \sim \frac{\pi(n-2)}{\sqrt{2}}\frac{\kappa}{d} \quad \text{as } d \to \infty,$$

where $T_{u_d}(f)(x) = \int_{S^{n-1}} u_d(\langle x, y\rangle) f(y)\, \mathrm{d}\mu(y)$.

This improves a result of Newman-Shapiro (1964) also generalised by Ragozin (1971)

Therefore, we get approximation guarantees for Lipschitz starbodies

Better than truncating the spherical harmonics series

$$f = \sum_{d=0}^{\infty} f_d$$

Corollary [M,M,V]: The polygauge body which approximates a convex body is convex as well.

Spherical harmonics

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A homogeneous polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d is called harmonic if $\Delta p = 0$.

Restrict to S^{n-1} → $\mathcal{H}_d$: spherical harmonics of degree d

$\mathbb{R}[S^{n-1}]$

$$\mathcal{H}_d = \mathbb{R}[S^{n-1}]_{\le d} \cap \mathbb{R}[S^{n-1}]^{\perp}_{\le d-1}$$

$$L^2(S^{n-1}, \mu) = \overline{\bigoplus_d \mathcal{H}_d}$$

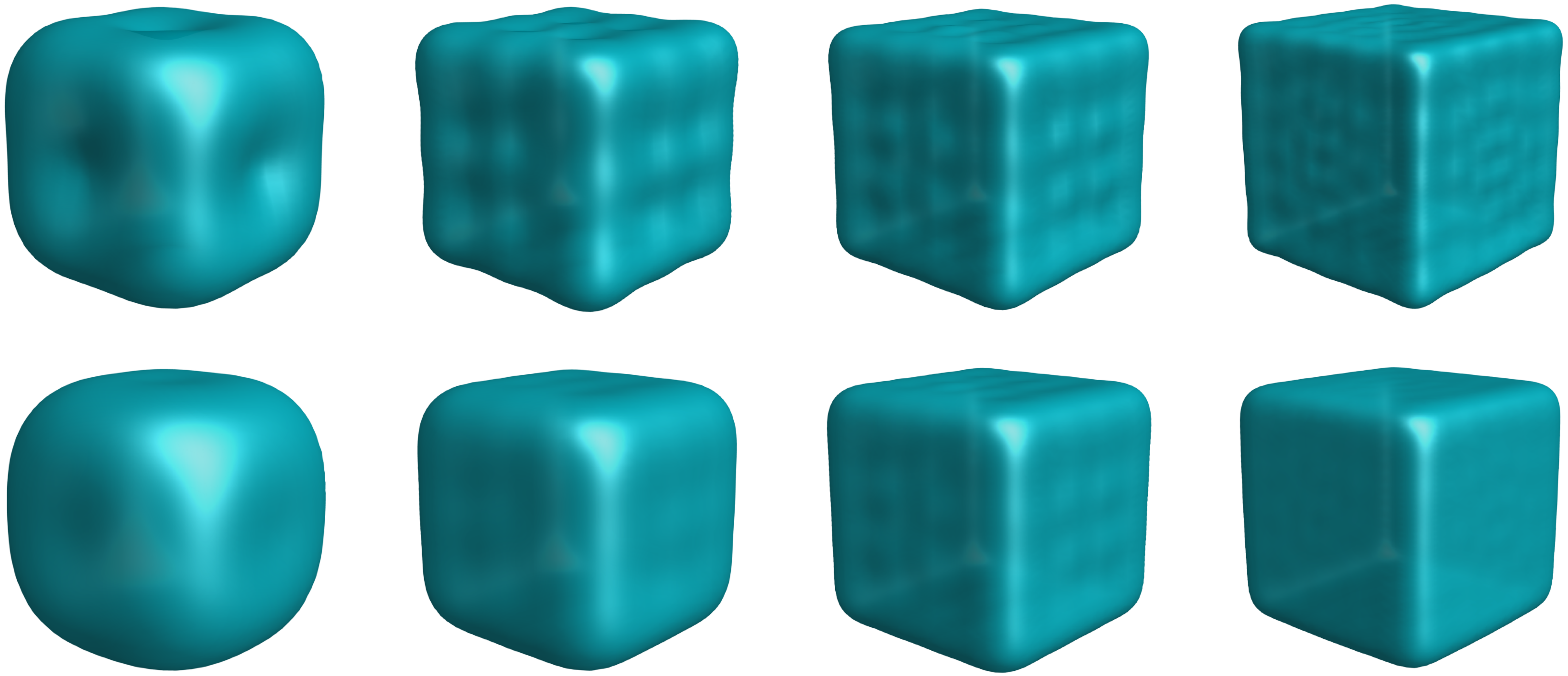
Each $f \in L^2(S^{n-1}, \mu)$ has a unique expression as $f = \sum_{d=0}^{\infty} f_d$ where $f_d \in \mathcal{H}_d$

To convince you with pictures

$d = 5$ | $d = 10$ | $d = 15$ | $d = 20$

Truncated harmonic series

$T_{u_d}(\rho_{\square})$



Can we do better?

* Theoretical ✓
* Theoretical constructive ✓
* Sharp approximation guarantees
* Computational

Can we do better if we approximate starbodies in other ways instead of with polynomials?

Kolmogorov N-width of a set $A \subset C(S^{n-1})$:

$$\mathscr{W}_N(A) = \inf_{\dim W \leq N} \sup_{a \in A} \mathrm{d}(a, W),$$

$W \subset C(S^{n-1})$ linear

i.e. best possible worst-case approximation error

Asymptotic optimality

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Theorem [M,M,V]: Lipschitz starbodies with Lipschitz constant at most κ on S^{n-1} cannot be approximated faster than

$$\frac{\kappa}{d} \quad \text{as } d \to \infty,$$

up to multiplicative constant, by bodies coming from any subspace of continuous function of the same dimension as the space of polynomials on S^{n-1} of degree at most d.

This extends an argument of Lorentz (1960)

Therefore, polystar bodies are asymptotically optimal approximators

Proofs ingredients:

* Gegenbauer polynomial
* Spherical harmonics
* Funk-Hecke formula

In practice

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- ✱ Theoretical ✓
- ✱ Theoretical constructive ✓
- ✱ Sharp approximation guarantees ✓
- ✱ Computational → How do I do it on my computer?

Use quadrature rules to compute $T_{u_d}(f)(x) = \int_{S^{n-1}} u_d(\langle x, y \rangle) f(y) \, \mathrm{d}\mu(y)$

ε-uniform convergence in poly(d) time → in fixed dimension n

∘ Python: https://github.com/ChiaraMeroni/polystar_bodies

Back to volume slices

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Wild? Tame?

The intersection body of L is the starbody IL with radial function

$$\rho_{IL}(x) = \mathrm{vol}(L \cap x^{\perp})$$

Lutwak (1988)
Towards the resolution of the Busemann-Petty problem

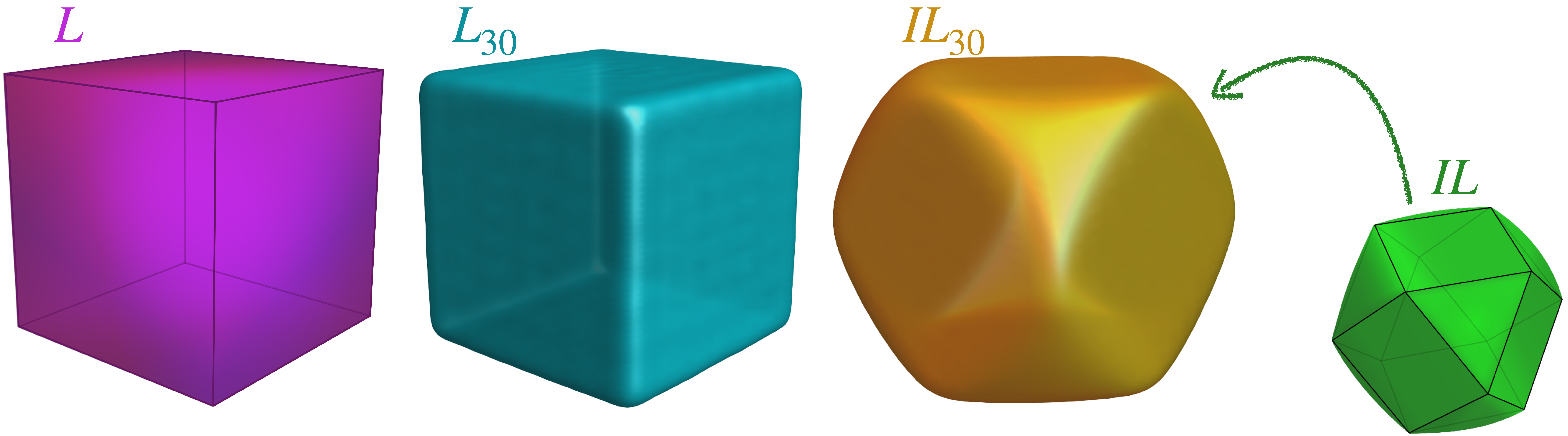
Largest volume slice of L ⟷ Maximum of ρ_{IL}

L polyradial body —Funk→ IL polyradial body

Computations

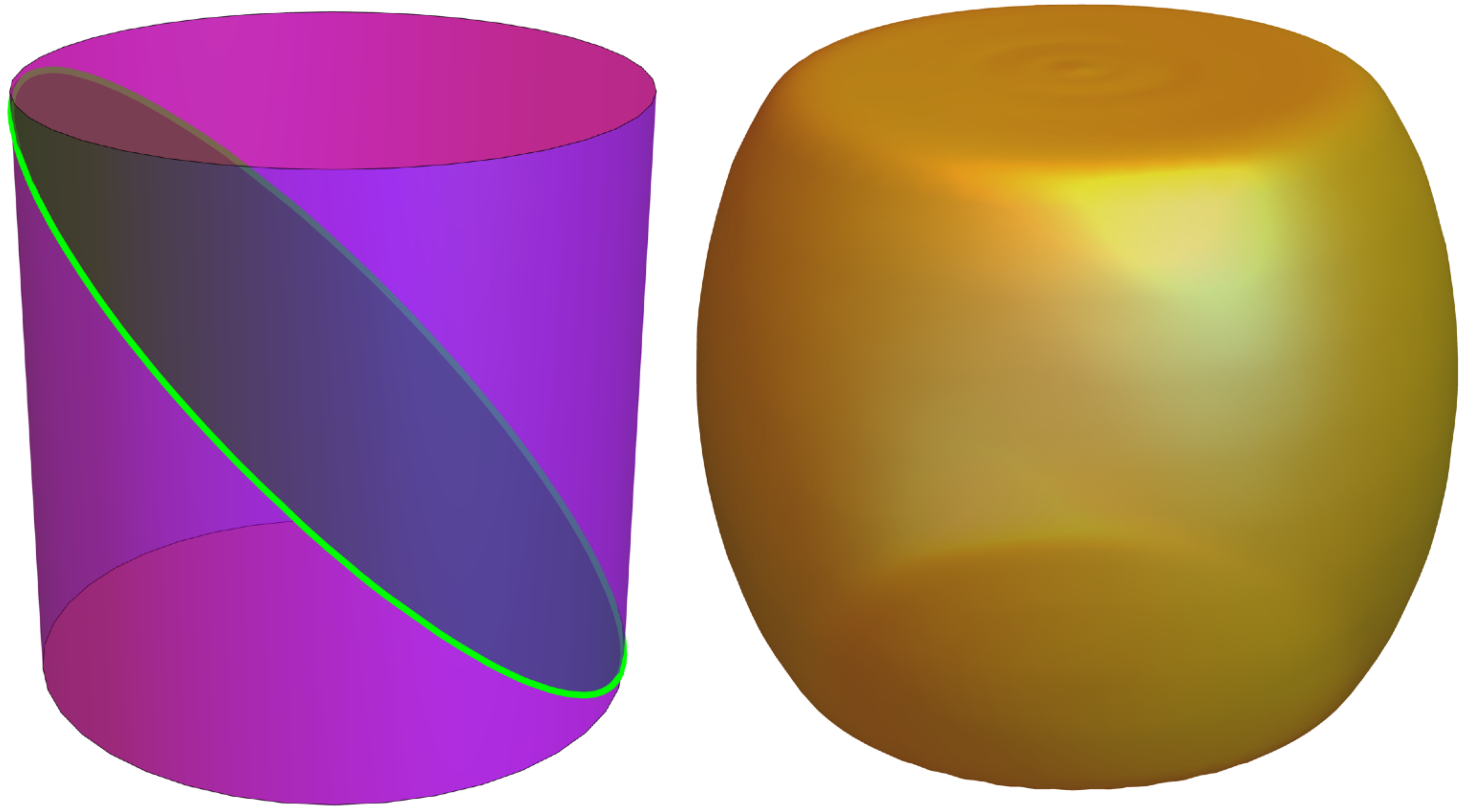
Computational pipeline:

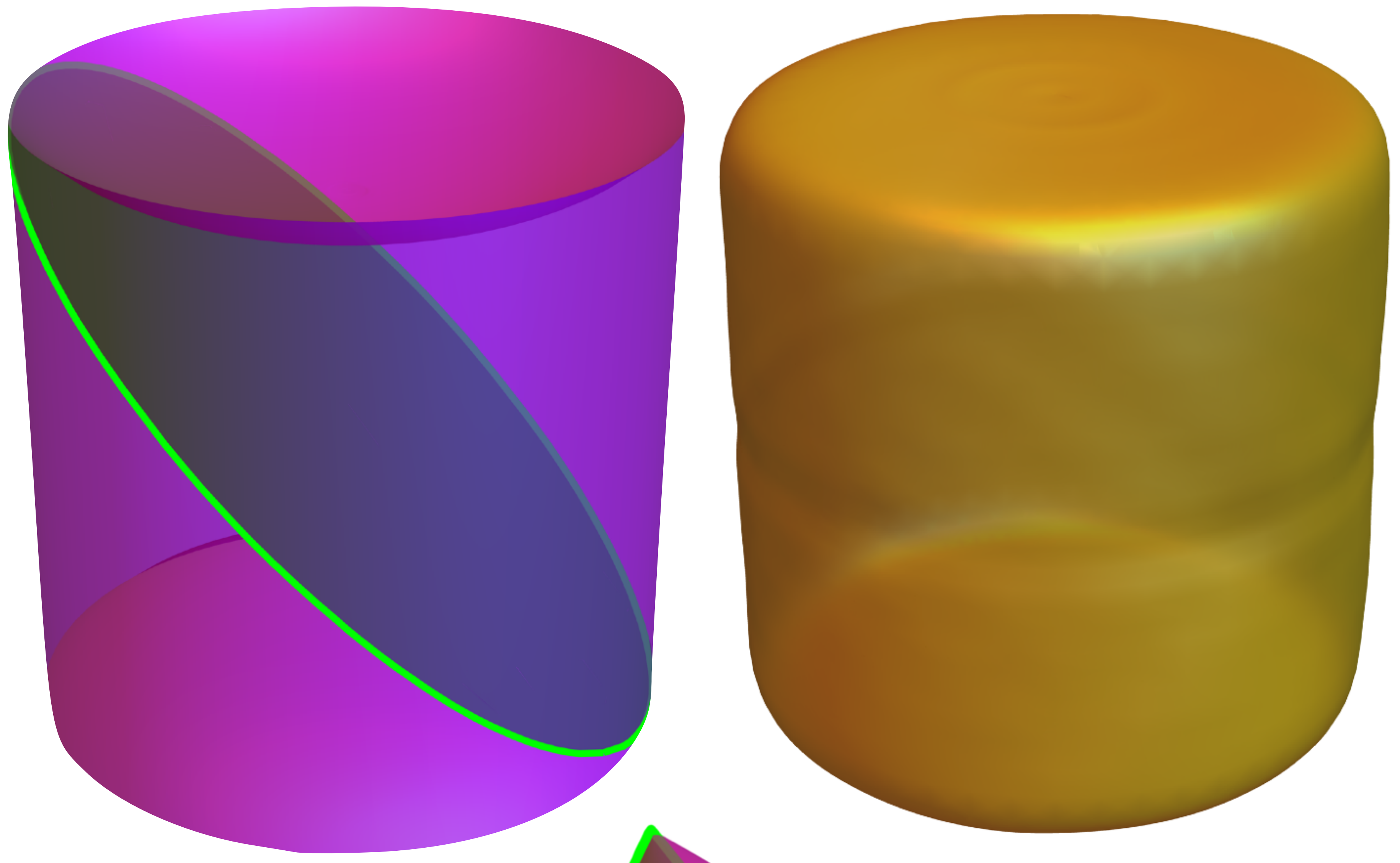
- Approximate L with a polyradial body L_d
- Compute the intersection body IL_d
- Find the maximum of the polynomial ρ_{IL_d} on S^{n-1}



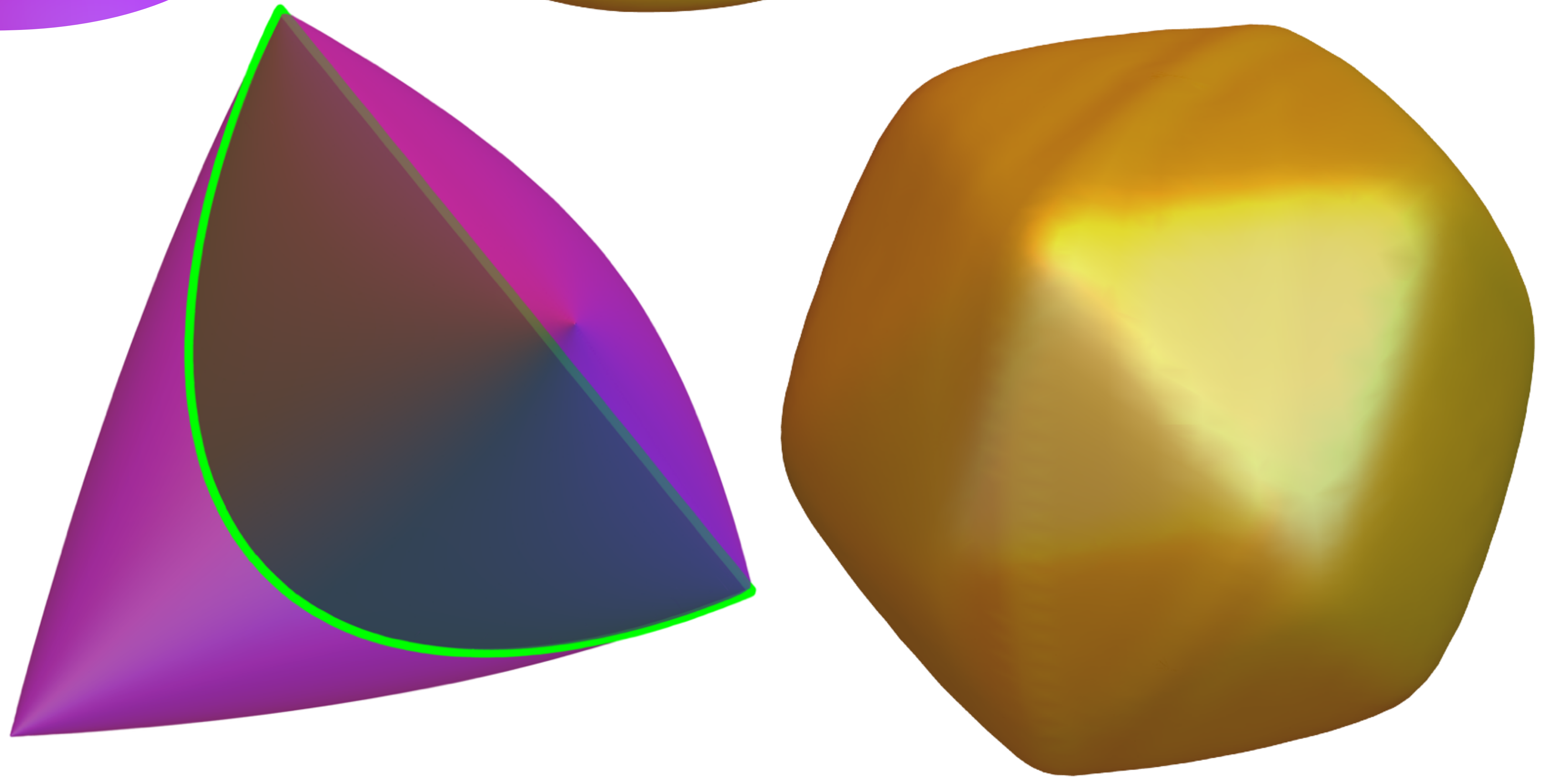
Computations

$d = 30$





L	numerical max	numerical direction	max	a direction
cube	5.4215	$(-0.7070, -0.7071, 0)$	$4\sqrt{2}$	$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$
cylinder	4.1184	$(0.6414, 0.2595, -0.7219)$	$\pi\sqrt{2}$	$(\frac{\cos t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
dented tin can	4.1713	$(0.6314, 0.3021, -0.7141)$	$\pi\sqrt{2}$	$(\frac{\cos t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
elliptope	3.7260	$(0.7089, -0.0026, -0.7052)$	$\frac{8\sqrt{2}}{3}$	$(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$



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ALGEBRAIC ASPECTS OF METRIC AND INTEGRAL GEOMETRY

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Thank you!