Transsserial trajectories of planar vector fields.

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40 ans déjà!

Our problem.

To study **formal vector fields** of dim 2:

$$\xi = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}, \quad \text{where } f,g \in \mathbb{R}[[x,y]]. \qquad (1)$$

via their **transserial solutions**

$$\begin{cases} x(t) & = & \gamma_1(t) \\ y(t) & = & \gamma_2(t) \end{cases}$$

where $\gamma = (\gamma_1, \gamma_2) \in \mathbb{T}^2$, $\varphi(0) = \psi(0) = 0$.

Our problem.

Our motivation: same problem for **dim 3 vector fields**

$$\xi = f(x,y,z)\frac{\partial}{\partial x} + g(x,y,z)\frac{\partial}{\partial y} + h(x,y,z)\frac{\partial}{\partial z}$$

where $f, g, h \in \mathbb{R}[[x,y,z]]$.

Work in progress...

Our problem

Formal counterpart of our work on 3-dim vector fields **definable in a polynomially bounded o-minimal structure over** $\mathbb{R}$:

Solutions of definable ODEs with regular separation and dichotomy interlacement versus Hardy

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Context and motivation.

General problem.

To describe the local dynamical behaviour of a vector field at a singular point.

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oscillating vs non-oscillating

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↘

non-oscillating

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Non-oscillating case $\rightsquigarrow$ to study the mutual behaviour of a pencil of trajectories

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interlacement vs separation

Known results.

Dimension 2.

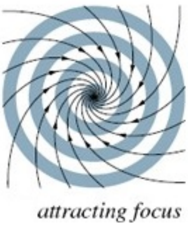
$$oscillation = spiralling$$

$$vs$$

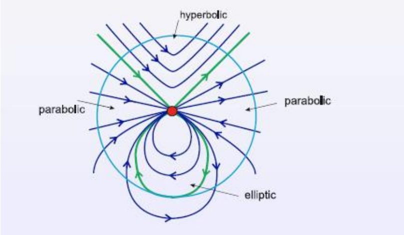
$$non\text{-}oscillation = has\ a\ tangent,$$

Known results.

Dimension 2.



VS



Known results.

Dimension 2.

oscillation = spiralling

vs

non-oscillation = has a tangent

+ *o-minimality* (Lion-Rolin 1998, Speissegger 1999,...)

Known results

Dimension 3, for a pencil having *iterated tangents* (Cano-Moussu-Sanz 2004):

oscillation = twisting around one *analytic* axis Γ_0

or

interlacing = twisting around one *formal* axis $\hat{\Gamma}_0$

+ *o-minimality* (Rolin-Sanz-Shäfke 2007)....

or

non-oscillation = separation by projection,

expected to *generate a Hardy field...*

Transseries and Hardy fields.

Grid-based transseries.

$\mathbb{T}_g$ = field obtained from $\mathbb{R}((t^{\mathbb{R}}))$ by closing under exp, (well-defined) log and taking **grid-based** series, i.e. series with support included in a set

$$\left\{\mathfrak{n}\,\mathfrak{m}_1^{\mathbb{N}}\cdots\mathfrak{m}_k^{\mathbb{N}}\right\} \text{ for some } \mathfrak{n}\,\mathfrak{m}_1,\ldots,\mathfrak{m}_k \in \mathfrak{M}.$$

Aschenbrenner–van den Dries–van der Hoeven (2017):

$$\mathbb{T}_g \preccurlyeq \mathbb{T}$$

*which is a **model complete** ordered valued differential field (H-field).*

Transseries and Hardy fields.

Hardy fields.

$\mathcal{H}$ = field of germs of $f : (a, +\infty) \to \mathbb{R}$ closed under derivation.

E.g.:

- via o-minimal structures;
- via non-oscillating solutions of ODE's.

Aschenbrenner–van den Dries–van der Hoeven (2023-25 preprint): *Maximal Hardy fields are $\equiv$ to $\mathbb{T}$, and therefore to $\mathbb{T}_g$.*

Transseries and differential equations

The natural valuation: v (or quasi-order $\preccurlyeq$) based on the archim. equiv. relation $\sim$.

Valuative properties:

1. **Strong l'Hospital's rule**: if $v(a) \neq 0$ and $v(b) \neq 0$, then $v(a) \geq v(b) \Leftrightarrow v(a') \geq v(b')$;
2. **Rule for the logarithmic derivative**: one has that $|v(a)| >> |v(b)| > 0$ if and only if $v(a'/a) < v(b'/b)$.
3. **Small derivation**: $v(a) \geq 0 \Rightarrow v(a') > 0$.

Transseries and differential equations

The natural valuation: v (or quasi-order $\preccurlyeq$) based on the archim. equiv. relation $\sim$.

Algebro-differential properties:

1. **Differential equations**: $\mathbb{T}$ is a real closed field closed under integration and logarithmic integration, so exp-log. It satisfies the **differential intermediate value property**.
2. **Compositional inverse**: for any $a(t) \in \mathbb{T}_{>\mathbb{R}}$, there is $b(t) \in \mathbb{T}_{>\mathbb{R}}$ such that $a(b(t)) = t$.

Transserial pencils

A **transserial curve** is the equivalence class of $\gamma = \gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \mathbb{T}^2 \setminus \{(0,0)\}$ up to *reparametrization*.

A **transserial trajectory** of a vector field ξ as in (1) is any transserial curve (at 0) such that any representative of the curve is tangent to ξ up to multiplication by some transseries.

Two curves γ, δ are **formally inseparable** if $\gamma \in A \Leftrightarrow \delta \in A$ for any semi-formal set A.

Integral pencil of ξ

$\mathcal{P} =$ a set of formally inseparable trajectories of ξ.

Our results.

Theorem (LGMPS)

Let ξ be a formal 2-dimensional vector field and $\mathcal{P}$ be a (non-empty) transserial pencil of ξ. Then ξ is not of centre-focus type and one of the following holds:

1. *$\mathcal{P}$ is formed of a single element γ, which is a formal curve.*
2. *There is a formal morphism $F : \mathbb{R}[[x, y]] \to (x, y)\mathbb{R}[[x, y]]$ s. t. $\mathcal{P} = F(\tilde{\mathcal{P}})$, where*
   - a *either $\tilde{\mathcal{P}} = \{(s, Cs^{\lambda}), C > 0\}$ for some $\lambda \in \mathbb{R}_{>0}$;*
   - b *or $\tilde{\mathcal{P}} = \{(s, Cs^{\mu} \exp(-1/ps^{p})) \text{ for some } \mu \in \mathbb{R}, p \in \mathbb{N}.$*

Moreover, there are but finitely many pencils of type (2).

Idea of proof.

Step 0 If O is a regular point: one unique formal power series solution (flow-box theorem).

$\rightsquigarrow$ *One pencil consisting of one formal curve.*

Idea of proof.

Assume O is a singular point.

Step 1 **Reduction of singularities** (Seidenberg 1968, and al.): *by blowing-up points finitely many times, one obtains ξ with only simple singularities.*

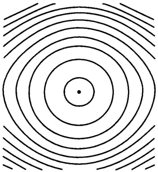
A vector field ξ has a **simple singularity** at a point *A* if the eigenvalues λ_1, λ_2 of its linear part at *A* satisfy:

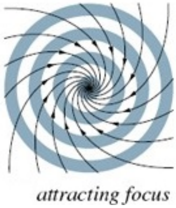
$$\lambda_1/\lambda_2 \notin \mathbb{Q}_{>0} \text{ and } \lambda_2 \neq 0.$$

Center-focus vector fields

A vector field ξ is of **centre-focus type** (also called **monodromic**) at 0 if and only if, in its (real) reduction of singularities there are:

- no dicritical components
- no non-corner singular points
- all corner-singular points are saddles, either hyperbolic or non-hyperbolic.





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Proposition

The vector field ξ is not of centre-focus type if and only if it has at least one transserial solution (and therefore at least one integral pencil).

Idea of proof

Assume O is a simple singular point not of center–focus type.

Step 2 **Reduction to formal normal forms (FNF)** (Poincaré, Dulac 1910's, and al.): *by a formal change of coordinates, reduction to the corresponding explicit simple vector field.*

E.g. the list from Il'Yashenko-Yakovenko's book (2007):

Idea of proof

Type	Conditions	Formal normal form
Nonresonant	$[\lambda_1 : \lambda_2] \notin \mathbb{Q}$ or $\lambda_1 = \lambda_2 \neq 0$	Linear
Resonant node	$[\lambda_1 : \lambda_2] = [r : 1]$, $r \in \mathbb{N}$, $r \geqslant 2$	$\dot{x} = rx + ay^r$, $\dot{y} = y$ $a \in \mathbb{C}$ formal invariant.
Resonant saddle (orbital)	$[\lambda_1 : \lambda_2] = -[p : q]$, $p, q \in \mathbb{N}$, not formally orbitally linearizable	$\dot{x} = -px$, $\dot{y} = qy(1 \pm u^r + au^{2r})$, $u = x^q y^p$, $r \in \mathbb{N}$, $a \in \mathbb{R}$ formal orbital invariants
Elliptic points (orbital)	$\lambda_{1,2} = \pm i\omega$, not formally orbitally linearizable	$\dot{x} = y \pm x(u^r + au^{2r})$, $\dot{y} = -x \pm y(u^r + au^{2r})$, $u = x^2 + y^2$, $a \in \mathbb{R}$ formal orbital invariant
Saddle-node (orbital classification)	$\lambda_1 \neq 0$, $\lambda_2 = 0$, formally isolated singularity	$\dot{x} = x$, $\dot{y} = \pm y^{r+1} + ay^{2r+1}$, $r \in \mathbb{N}$, $a \in \mathbb{R}$ formal orbital invariants

Idea of proof

Step 3 **Explicit resolution in terms of transseries** (LGMPS): *we compute explicit solutions by integration and resolution of implicit equations.*

Example: if the point O is a node i.e. $\lambda := \lambda_1/\lambda_2 \in (\mathbb{R} \setminus \mathbb{Q})_{>0}$ and

$$\xi = -x\frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y},$$

then it has 3 transserial pencils (up to reparametrization $t \leftrightarrow \log(t)$, written with $s := 1/t$ near 0^+):

$$\begin{array}{lcl} \mathcal{P}_+ & := & \{\gamma_C = (s, Cs^\lambda), \mid C \in \mathbb{R}_{>0}\} \\ \mathcal{P}_- & := & \{\gamma_C = (s, Cs^\lambda), \mid C \in \mathbb{R}_{<0}\} \\ \mathcal{P}_0 & := & \{\gamma_0 = (s, 0)\} \end{array}$$

Idea of proof

Step 4 **Going back to the original coordinates:** *blow-down + inversion of the normal form coordinates + inversion of transseries*

Example: consider a vector field in FNF coordinates (x, y)

$$\xi = -x\frac{\partial}{\partial x} - \lambda y\frac{\partial}{\partial y},$$

and suppose that the inversion $F(a, b)$ of this FNF change of coordinates is:

$$x = a - a^2 + 2ab \qquad \dot{x} = (1 - 2a + 2b)\dot{a} + (2a)\dot{b}$$

$$y = b - 3a^2 b \qquad \dot{y} = (-6ab)\dot{a} + (1 - 3a^2)\dot{b}$$

Example continued

The corresponding vector field is:

$$(a + 2a^3 - 2(\lambda + 1)a^b + \cdots)\frac{\partial}{\partial a} + (\lambda b + 6a^2 b + \cdots)\frac{\partial}{\partial b}$$

or equivalently for $a \neq 0$:

$$(a^3 + 2a^5 - 2(\lambda + 1)a^4 b + \cdots)\frac{\partial}{\partial a} + (\lambda a^2 b + 6a^4 b + \cdots)\frac{\partial}{\partial b}$$

Example continued

By blow-down, e.g. by:

$$a = u \qquad \dot{x} = \dot{u}$$

$$b = \frac{v}{u} \qquad \dot{y} = \frac{1}{u}\dot{v} - \frac{v}{u^2}\dot{u}$$

the vector field corresponds in the original coordinates (u, v) to:

$$(u^3 - 2(\lambda+1)u^3 v + 2u^5 + \cdots)\frac{\partial}{\partial u} + (\lambda uv + 6u^3 v + \cdots)\frac{\partial}{\partial v}$$

Example continued

The initial coordinates are:

$$
\begin{aligned}
u =& x + x^2 + 2xy + \cdots \\
v =& xy + x^2 y + 2xy^2 + \cdots
\end{aligned}
$$

The pencil $\mathcal{P}_+ := \{\gamma_C = (s, Cs^\lambda), | \ C \in \mathbb{R}_{>0}\}$ in coordinates (x, y) above is given by:

$$
\begin{aligned}
u =& s + s^2 + 2Cs^{1+\lambda} + \cdots \\
v =& Cs^{1+\lambda} + Cs^{2+\lambda} + C^2 s^{1+2\lambda} + \cdots
\end{aligned}
$$

Example finished

Finally, if one dares to invert transseries e.g. s in terms of u:

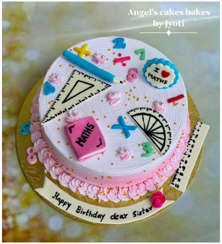
$$s = u + u^2 + 2Cu^{1+\lambda} + \cdots$$

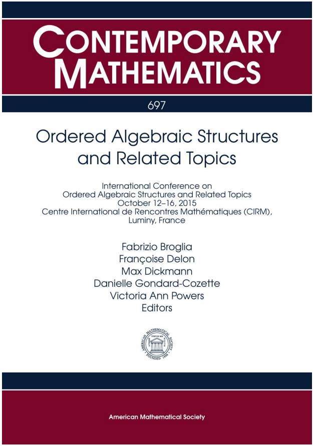
one obtains by composition a pencil, e.g. for $\mathcal{P}_+$:

$$\mathcal{P}_+ := \{\gamma_C = (u, F(u, Cs^\lambda)| \ C \in \mathbb{R}_{>0}\}.$$

Thank you for your attention...

... and Happy DDG 40th BIRTHDAY!!

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