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Composition of transseries, monotonicity, and analyticity

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Supported by EPSRC EP/T018461/1

DDG40, Banyuls-sur-Mer 4–8 August 2025

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Remember Taylor series

Fix a small interval $I := (-\varepsilon, \varepsilon) \subseteq \mathbb{R}$ and take the (infinitely) differentiable functions $\mathcal{C}^\infty(I)$ on it.

The Taylor expansion of a function $f \in \mathcal{C}^\infty(I)$ is the power series

$$\mathrm{T}(f) := f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots \in \mathbb{R}[[x]].$$

Fact. $\mathbb{R}[[x]]$ has the following structure:

- ordered ring: $a_m x^m + a_{m+1}x^{m+1} + \ldots$ is positive if and only if $a_m > 0$; sum and product are 'obvious';
- differential ring with $\frac{d}{dx}$: extend $(x^n)' = nx^{n-1}$ (uniquely) by *strong linearity*;
- composition: if $Q \in x\mathbb{R}[[x]]$, extend $x^n \circ Q = Q^n$ by strong linearity. Chain rule: $(P \circ Q)' = (P' \circ Q)Q'$.

Exercise. The map T is a morphism of differential rings with composition.

- T is only injective on analytic functions $\mathcal{C}^\omega(I)$ or other *quasianalytic* classes (by definition!).
- $\sqrt{x}$ is differentiable on $I^+ := (0, \varepsilon)$ but is not captured by T.
- $e^{-\frac{1}{x^2}}$ is in $\ker(\mathrm{T})$; $e^{-\frac{1}{x}}$ has different limits at 0^+ and 0^-.

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Extending Taylor series

$\mathbb{T}$ does not capture roots, $e^{-\frac{1}{x^2}}, e^{-\frac{1}{x}}$.

- To work at '0^+', use $I^+ = (0, \varepsilon)$; conventionally, actually $(a, +\infty)$ with $t = x^{-1}$. (Now please forget x.)
- To do roots: include new monomials t^r for $r \in \mathbb{R}$.
  Now $\mathbb{R}((t^{\mathbb{R}}))$ consists of series *indexed by ordinals*: $\sum_{i<\alpha} a_i t^{r_i}$, where $i < j$ implies $t^{r_i} > t^{r_j}$.
- For the exponentials, add *more monomials*, such as $e^{-t}, e^{-t^2}, \log(t), e^{t^3+\log(t)}$. Require $e^t > t^r$.

Let us close $\mathbb{R}((t^{\mathbb{R}}))$ under 'infinite sums, exp, log'. Build inductively an ordered group of monomials $\mathfrak{M}_\alpha$ and partially defined maps $\log_{\mathbb{J}} : \mathfrak{M}_\alpha \to \mathbb{J}_\alpha$, $\exp_{\mathbb{J}} : \mathbb{J}_\alpha \to \mathfrak{M}_\alpha$, where $X_\alpha := \mathbb{R}((\mathfrak{M}_\alpha))$, $\mathbb{J}_\alpha := \mathbb{R}[[\mathfrak{M}_\alpha^{>1}]]$.

1. Base step: $\mathfrak{M}_0 := t^{\mathbb{R}}$. Here $X_0 = \mathbb{R}((\mathfrak{M}_0)) = \mathbb{R}((t^{\mathbb{R}}))$ and $\mathbb{J}_0 = \mathbb{R}[[\mathfrak{M}_0^{>1}]]$.
2. For $n < \omega$: $\mathfrak{M}_{n+1} := \exp_{\mathbb{J}}(\mathbb{J}_n) \cdot \log^{\circ n}(t)^{\mathbb{R}} \cdot \log^{\circ(n+1)}(t)^{\mathbb{R}}$. Define $\exp_{\mathbb{J}}, \log_{\mathbb{J}}$ the 'obvious' way.
3. For $\alpha \geq \omega$: $\mathfrak{M}_{\alpha+1} := \exp_{\mathbb{J}}(\mathbb{J}_\alpha)$, and $\mathfrak{M}_\alpha := \bigcup_{\beta<\alpha} \mathfrak{M}_\alpha$ if α limit.

Finally $\exp(f) := \exp_{\mathbb{J}}(f_{>1}) \cdot e^{f_{=1}} \cdot (\mathbb{T}(\exp) \circ f_{<1}) = \exp(f_{>1} + f_{=1} + f_{<1})$.

$\bigcup_\alpha X_\alpha$ is the Field $\mathbb{R}\langle\langle t \rangle\rangle$ of omega-series (a proper class!), $\bigcup_{n<\omega} X_n$ is the field $\mathbb{T}$ of LE-series.

(References: too many to fit into the slide.)

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From the first slide: T is a morphism for $<, +, \cdot, \frac{d}{dx}, \circ$, and injective on (quasi)analytic functions.

Facts/exercises. $\mathbb{R}\langle\langle t\rangle\rangle$ has the following structure.

- ordered ring/field: just because it is a field of generalised power series;
- derivation $\frac{d}{dt}$: *unique* strongly linear extension with $(\exp(f))' = \exp(f) \cdot f'$;
- composition: *unique* strongly linear extension of $t \circ g = g$ with $\exp(f) \circ g = \exp(f \circ g)$ (where $g > \mathbb{R}$); the chain rule holds $(f \circ g)' = (f' \circ g) \cdot g'$.

To be honest: uniqueness is trivial, but existence is a combinatorial headache.

Note. There is no *canonical* embedding function like T: constructing embeddings into $\mathbb{R}\langle\langle t\rangle\rangle$ is a non-trivial task. (Random references: van der Hoeven '09 for some Hardy fields, ADH '24 for $\mathbb{R}_{\mathrm{Pfaff}}$, Rolin-Servi-Speissegger '24, Freni '24 for some o-minimal structures.)

In fact, Dulac's problem was recently re-declared open by Ilyashenko, because the 'embedding' used in the proof may not be injective (or at least, the proof has a gap).

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Recall: $\mathbb{R}\langle\langle t\rangle\rangle$ is *generated* by t, so derivation and composition are uniquely determined.

Proposition. $(\mathbb{R}\langle\langle t\rangle\rangle, <, +, \cdot, \frac{d}{dt})$ is an '*H*-field' with constant field $\mathbb{R}$. Almost everything has an antiderivative.

$(\mathbb{R}\langle\langle t\rangle\rangle, <, +, \cdot, \frac{d}{dt})$ is *almost* a model of the theory of '*H*-closed fields with small derivation', which is the theory of $\mathbb{T}$, model-complete after adding the valuation to the language, NIP, distal (ADH '17), and it is also the theory of all maximal Hardy fields (ADH '24).

Exercise. Axiomatise the theory of $\mathbb{R}\langle\langle t\rangle\rangle$ as an ordered valued differential field (questionable cost/benefit).

$(\mathbb{R}\langle\langle t\rangle\rangle^{>\mathbb{R}}, \circ)$, even without the field structure, is a different beast. This is a highly non-abelian group; *hyperseries* are a little better, with only three conjugacy classes (Bagayoko). Ask Vincent Bagayoko about 'growth order groups'.

But what about basic properties? For instance: surely the map $g \mapsto f \circ g$ is strictly increasing when $f' > 0$?

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Monotonicity

Theorem (monotonicity; M., apparently). For all $f, x, y \in \mathbb{R}\langle\langle t \rangle\rangle$ with $y > x > \mathbb{R}$, we have $f \circ x < f \circ y \Leftrightarrow f' > 0$.

Not completely obvious! Naive induction (some details slightly off for simplicity):

- Suppose monotonicity holds in X_α (easy for $\alpha = 0$).
- Let $f = \sum_i r_i e^{\gamma_i} \in X_{\alpha+1}$, where $\gamma_i \in \mathbb{J}_\alpha \subseteq X_\alpha$. Write $f \circ y - f \circ x = \sum_i r_i(e^{\gamma_i} \circ y - e^{\gamma_i} \circ x)$.
- Verify by induction that $r_0(e^{\gamma_0 \circ y} - e^{\gamma_0 \circ x})$ *dominates* all other terms, hence monotonicity (!!).

But (!!) is only easy for $\gamma_i > 0$, in which case $|\gamma_0| > |\gamma_i| > 0$, thus $\gamma_i \circ y - \gamma_i \circ x$ is smaller than $\gamma_0 \circ y - \gamma_0 \circ x$. When $\gamma_i < 0$, then $|\gamma_i|$ may well be bigger than γ_0, and that puts the inequalities the wrong way around.

Pragmatic answer: use a stronger inductive hypothesis, a sort of weak mean value theorem. Assume for $\gamma \in \mathbb{J}_\alpha$:

- if $\gamma \circ y - \gamma \circ x \in O(1)$, then $\gamma \circ y - \gamma \circ x \in O((\gamma' \circ x) \cdot (y - x))$;
- if $1 \in O(\gamma \circ y - \gamma \circ x)$, then $1 \in O((\gamma' \circ x) \cdot (y - x))$.

Exercise. Prove by induction that the above condition holds in $\mathbb{R}\langle\langle t \rangle\rangle$.

To deduce monotonicity: if $f > \mathbb{R}$, do $\log_n(t) \circ f \circ \exp_k(t)$ to reduce to $f = \omega + \varepsilon$. Verify that $f \circ y - f \circ x \sim y - x$. Other cases follows by algebraic manipulations.

Question. Can this be adapted to hyperseries?

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Consider $\delta \mapsto f \circ (t+\delta)$ (more generally $f \circ (x+\delta)$). Is it analytic? What do we mean by that?

Proposition (Weak Taylor, M.). Let $f, x, \delta \in \mathbb{R}\langle\langle t \rangle\rangle$ with $x > \mathbb{R}$, $\delta \in o(x)$, $f \notin O(1) \setminus o(1)$. If $(f^\dagger \circ x) \cdot \delta \in o(1)$, then

$$f \circ (x+\delta) = f \circ x + (f' \circ x) \cdot \delta + \frac{f'' \circ x}{2!} \cdot \delta^2 + \ldots + \frac{f^{(n)} \circ x}{n!} \cdot \delta^n + O((f^{(n+1)} \circ x) \cdot \delta^{n+1})$$

Exercise. Prove the Proposition by induction, using monotonicity (and formulas relating $f^{(n)}$ to $(f^\dagger)^n \cdot f$).

Exercise. Prove that $f' = \lim_{\delta \to 0} \frac{f \circ (t+\delta) - f \circ t}{\delta}$ (was in Berarducci-M, but only for δ surreal).

The infinite sum $\sum_n \frac{f^{(n)} \circ x}{n!} \delta^n$ can be in $\mathbb{R}\langle\langle t \rangle\rangle$ as well. When does it coincide with $f \circ (x+\delta)$?

Detour. $x \mapsto f \circ x$ makes sense also if x lives in a *different* field, e.g. the surreals. For δ surreal, the equality holds when δ infinitesimals w.r.t. *all of* $\mathbb{R}\langle\langle t \rangle\rangle$ and a bit more (Berarducci-M.). But what is the 'radius of convergence'?

Theorem (Strong Taylor, Bagayoko-M., ~Schmeling). Let $f, x, \delta \in \mathbb{R}\langle\langle t \rangle\rangle$ as before. If $(\mathfrak{m}^\dagger \circ x) \cdot \delta \in o(1)$ for every monomial $\mathfrak{m}$ of f, then

$$f \circ (x+\delta) = \sum_n \frac{f^{(n)} \circ x}{n!} \delta^n. \qquad \text{(radius of convergence is essentially optimal)}$$

Strong Taylor, the scary version

Let $\mathbb{A}$ a 'differential pre-logarithmic H-field' satisfying

$$\mathfrak{m}^\dagger \in O(t^{-1}) \Rightarrow (\operatorname{supp} \mathfrak{m}')^\dagger \in O(t^{-1}) \quad \text{and} \quad t^{-1} \in o(\mathfrak{m}^\dagger) \Rightarrow (\operatorname{supp} \mathfrak{m}')^\dagger = \Omega(\mathfrak{m}^\dagger)$$

for some fixed t.

Let $\mathbb{B}$ be a field of generalised power series and $\triangle : \mathbb{A} \to \mathbb{B}$ a strongly linear algebra morphism.

Theorem (General Convergence Theorem, Bagagyoko-M.). Let $f \in \mathbb{A}$, $\delta \in \mathbb{B}$ with $\delta \in o(\triangle(t))$, $\triangle(\mathfrak{m}^\dagger)\delta \in o(1)$ for all $\mathfrak{m}$ of f. Then the following expression is summable:

$$\sum_n \frac{\triangle(f^{(n)})}{n!}\delta^n.$$

This is then applied to $\triangle(f) = f \circ x$ either within $\mathbb{R}\langle\langle t \rangle\rangle$, or for instance with $f \in \mathbb{R}\langle\langle t \rangle\rangle$ and $x \in \mathbf{No}$.

Uniqueness of the composition in $\mathbb{R}\langle\langle t \rangle\rangle$ guarantees that the sum coincides with $f \circ (x + \delta)$.

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Monotonicity and weak Taylor for normalisation

Vague classification idea: two functions (or germs) are 'similar' if they become the same after a 'controlled' change of variables.

Example. Given $r, s \in \mathbb{R}^{>1}$, the functions t^r and t^s are conjugate to tr and $t + \log(r)$, ts and $t + \log(s)$, and $t + 1$.

- $\log(t) \circ t^r \circ \exp(t) = tr$; likewise for t^s;
- $\log(t) \circ (tr) \circ \exp(t) = t + \log(r)$; likewise for ts;
- $\frac{t}{\log(r)} \circ (t + \log(r)) \circ (t \log(r)) = t + 1$; likewise for $t + \log(s)$.

Question (Bagayoko?). What are the conjugacy classes of $\mathbb{R}\langle\langle t \rangle\rangle$ by composition? (Some are $\exp^{\circ n}(t)$ for $n \in \mathbb{Z}$.)

More concrete problem: classify 'Dulac series' by change of variables that are 'tangent to the identity'.

$$f = c_0 t + \sum_i t^{\nu_i} P_i(\log(t)), \quad \phi = t + \varepsilon, \quad \phi \circ f \circ \phi^{-1}?$$

The above is the 'hyperbolic case' at $t \to +\infty$ (see Mardešić-Resman-Rolin-Županović).

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Normalisation of hyperbolic omega series

Let $f = ct + \varepsilon \in \mathbb{R}\langle\langle t \rangle\rangle$ with $c > 0, c \neq 1, \varepsilon \in o(t)$.

Theorem (Peran-Resman-Rolin-Servi). Suppose that the monomials of ε belong to $t^{\mathbb{R}} \cdot \log(t)^{\mathbb{R}} \cdots \log^{\circ n}(t)^{\mathbb{R}}$, and $c < 1$. Then there is a unique $\phi = t + \delta$, with $\delta \in o(t)$, (with monomials is the same group!) such that

$$\phi^{-1} \circ f \circ \phi = f \text{ with the monomials (strictly) smaller than } \frac{t}{\log(t) \cdots \log^{\circ n}(t)} \text{ truncated away.}$$

For example: $f = \frac{1}{2}t + \frac{t}{\sqrt{\log(t)}} + \frac{t}{\log^2(t)} + \log(t) + t^{-1}$ can be conjugated to $f = \frac{1}{2}t + \frac{t}{\sqrt{\log(t)}}$, but no shorter.

Theorem (M.-Peran-Rolin-Servi). If the monomials of ε belong to a 'good' group $\mathfrak{M}$, there is a unique $\phi = t + \delta$ with monomials in $\mathfrak{M}$ and $\delta \in o(t)$ such that

$$\phi^{-1} \circ f \circ \phi = f \text{ with the monomials smaller than } t^2 \cdot \inf \Psi(\mathfrak{M}) \text{ truncated away.}$$

($\inf \Psi(\mathfrak{M})$ is the unique monomial/cut that does not admit asymptotic integration in $\mathfrak{M}$.)

Weak Taylor yields approximations of $f \circ \phi$ with sufficiently precise error terms. We e.g. clarify that the terms $\mathfrak{m}$ that can be conjugated away are exactly the ones such that $\frac{\mathfrak{m}}{t^2}$ admits an asymptotic integral.

Thanks!

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