Arithmetic of cuts in ordered abelian groups and of ideals over valuation rings

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A problem to be solved

The following problem came up in the papers

S.D. Cutkosky – K – A. Rzepka: *On the computation of Kähler differentials and characterizations of Galois extensions with independent defect*, Mathematische Nachrichten **298** (2025), 1549–1577

and

K – S.D. Cutkosky: *Kähler differentials of extensions of valuation rings and deeply ramified fields*, submitted; available at `https://www.fvkuhlmann.de/Fvkprepr.html`.

Take a valuation ring $\mathcal{O}$ and two $\mathcal{O}$-ideals U, V. Compute the annihilator of U/V or of U/UV.

The annihilator of an $\mathcal{O}$-module M is

$$\operatorname{ann} M \;:=\; \{a \in \mathcal{O} \mid aM = 0\}\,.$$

This problem was the reason for writing my joint paper *Arithmetic of cuts in ordered abelian groups and of ideals over valuation rings*, submitted; available at
`https://www.fvkuhlmann.de/Fvkprepr.html`,
with Katarzyna Kuhlmann on which I will now report.
It uses and extends parts of the unpublished manuscript *Invariance group and invariance valuation ring of a cut*, also available at the URL given above.

A quick answer, but...

It is easy to check that for any two $\mathcal{O}$-ideals I_1 and I_2,

$$I_2 : I_1 \;:=\; \{a \in K \mid aI_1 \subseteq I_2\} \tag{1}$$

is the largest solution J for the inequality

$$I_1 J \subseteq I_2 . \tag{2}$$

Therefore,

$$\operatorname{ann} I_2/I_1 \;=\; I_2 : I_1 .$$

However, if we have an explicit description of the ideals I_1 and I_2, as is the case in the problem I have mentioned, then we want a corresponding explicit computation of $I_2 : I_1$.

Ribenboim's contribution

In his article

P. Ribenboim: *Sur les groupes totalement ordonnés et l'arithmétique des anneaux de valuation*, Summa Brasil. Math. **4** (1958), 1–64,

Paulo Ribenboim deals with the existence and computation of solutions of equations $I_1 J = I_2$. He uses a reduction of the problem to a corresponding problem about ordered abelian groups, which I will describe now.

Let us first set the notation. We denote by K the quotient field of $\mathcal{O}$, by v the valuation on K, and by vK its value group. When we talk of $\mathcal{O}$-ideals, we will always assume them to be nonzero, and we also include fractional ideals. We say that $S \subsetneq vK$ is a final segment of vK if for every $\alpha \in S$ and every $\gamma \in vK$ with $\gamma \geq \alpha$, it follows that $\gamma \in S$.

From ideals to final segments

The function

$$v : I \mapsto vI := \{vb \mid 0 \neq b \in I\} \tag{3}$$

is an order preserving bijection from the set of all $\mathcal{O}$-ideals onto the set of all final segments of vK: $J \subseteq I$ holds if and only if $vJ \subseteq vI$ holds. The inverse of this function is the order preserving function

$$S \mapsto I_S := (a \in K \mid va \in S) . \tag{4}$$

Further, the function (3) is a homomorphism from the multiplicative monoid of $\mathcal{O}$-ideals onto the additive monoid of final segments:

$$v\,I\,J = vI + vJ \quad \text{and} \quad I_{S+S'} = I_S\,I_{S'} . \tag{5}$$

Solving equalities for final segments

Via the function (3), the inequality $I_1 J \subseteq I_2$ is translated to the inequality

$$S_1 + T \subseteq S_2 \tag{6}$$

of final segments in vK. This inequality has the largest solution

$$S_2 -_{\text{fs}} S_1 := \{\alpha \in vK \mid \alpha + S_1 \subseteq S_2\}\,. \tag{7}$$

But again, if we have an explicit description of the final segments S_1 and S_2, then we want a corresponding explicit computation of $S_2 -_{\text{fs}} S_1$.

Note that $T = S_2 -_{\text{fs}} S_1$ may not be a solution of the equation $S_1 + T = S_2$, in which case the equation has no solution at all.

Take an ordered abelian group Γ. The Dedekind cuts in Γ are uniquely determined by their upper cut sets, which are nonempty final segments of Γ. One way to introduce an addition on the collection of Dedekind cuts is to add their upper cut sets. However, this will in general yield a monoid, but not a group. There may even be nontrivial idempotents, derived from convex subgoups.

Therefore, equations $S_1 + T = S_2$ can have several solutions, or no solutions at all. One can thus not expect a subtraction in the usual sense to exist. There have been several attempts at defining subtractions, but the above turns out to be the one that solves inequalities of the form $S_1 + T \subseteq S_2$.

Ribenboim's contribution

In his article, Ribenboim determines when the equation $S_1 + T = S_2$ has a solution, and if so, determines the number of possible solutions. However, his approach and the presentation of solutions are needlessly complicated. The reason could be called a variant of the *horror vacui*. I have observed this also in another paper of Ribenboim, from 1967, in which he computes higher ramification ideals. There, he unnecessarily restricts the scope to rank one valuations, that is, valuations with archimedean value group, but it is easy to generalize his approach to general rank. If the rank of the value group is larger than one, it means that it cannot be embedded in the ordered additive group of the reals, and so infima will not always exist.

Ribenboim does not realize that one can substitute them by final segments; instead, he works with embeddings of ordered abelian groups in Hahn products.

Some notation and basic facts

For $\gamma \in \Gamma$, we set

$$\gamma^- := \{\alpha \in \Gamma \mid \alpha \geq \gamma\} .$$

We call a final segment principal if it is of the form γ^-. We will denote the set of all nonempty final segments of $(\Gamma, \leq)$ by

$$\Gamma^\uparrow .$$

This set is totally ordered by inclusion. The function

$$\Gamma \ni \gamma \mapsto \gamma^- \in \Gamma^\uparrow \tag{8}$$

is an embedding of $(\Gamma, \leq)$ in $(\Gamma^\uparrow, \subseteq)$. It is onto if and only if $\Gamma \simeq \mathbb{Z}$. In this case, every equation $S_1 + T = S_2$ has a unique solution.

Some notation and basic facts

Take final segments of Γ. Then:

1) The sum of two final segments is again a final segment.

2) The sum of two principal final segments is again a principal final segment: $\alpha^- + \beta^- = (\alpha + \beta)^-$.

3) The sum of two final segments of which at least one is nonprincipal is a nonprincipal final segment.

4) Consequently, if S_1 and S_2 are final segments of Γ and S_2 is principal but S_1 is not, then the equation $S_1 + T = S_2$ has no solution.

Invariance groups

A key to the answers for the above questions is the notion of *invariance group* of a final segment, which also appears in Ribenboim's article, in a somewhat disguised form. Take an arbitrary ordered abelian group. For a final segment S of Γ, we define its invariance group to be

$$\mathcal{G}(S) \ := \ \{\gamma \in \Gamma \mid S + \gamma = S\} \, ,$$

which is always a convex subgroup of Γ. It sheds light on the important role that convex subgroups play in ordered groups of general rank for solving inequalities of the form $S_1 + T \subseteq S_2$. The case of ordered groups of rank one, which have no proper nontrivial convex subgroups, is significantly easier.

For background and applications of invariance groups, see K: *Selected methods for the classification of cuts and their applications*, Proceedings of the ALANT 5 conference 2018, Banach Center Publications **121** (2020), 85–106.

An ordered group Γ has rank one, i.e., is archimedean, if and only if every final segment $S \subsetneq \Gamma$ has trivial invariance group $\mathcal{G}(S) = \{0\}$. Therefore, the next proposition applies in particular (but not only) to archimedean ordered groups.

Given a final segment S in Γ, we set

$$\widehat{S} := \begin{cases} S \cup \{\gamma\} & \text{if } S \text{ has infimum } \gamma \text{ in } \Gamma\text{, and} \\ S & \text{otherwise.} \end{cases}$$

Then $\widehat{S}$ is the closure of S in Γ in the order topology. We also denote it by $S\widehat{\ }$.
We denote by S^c the complement of S in Γ.

The case of trivial invariance groups

Proposition

Take final segments S_1 and S_2 of Γ. Assume that

$$\mathcal{G}(S_1) = \mathcal{G}(S_2) = \{0\}$$

and that S_2 is nonprincipal if S_1 is nonprincipal.

Then $T = S_2 -_{\rm fs} S_1$ is a solution (and hence the largest) of the equation $S_1 + T = S_2$, and we have

$$S_2 \ -_{\rm fs} \ S_1 \ = \begin{cases} S_2 - \alpha & \text{if } S_1 = \alpha^- \text{ for some } \alpha \in \Gamma, \\ (S_2 - S_1^c)^\wedge & \text{if } S_1 \text{ is nonprincipal.} \end{cases} \tag{9}$$

Preparations for the case of arbitrary rank

For the image of an element $\gamma \in \Gamma$ under the canonical epimorphism

$$\varphi_H : \Gamma \to \Gamma/H ,$$

i.e., the coset $\gamma + H$, we prefer to write γ/H since we also have to deal with subsets of the form $\gamma + H$ in Γ. With this notation, $M/H = \{\gamma/H \mid \gamma \in M\}$ for every subset $M \subseteq \Gamma$. Since the epimorphism preserves $\leq$, we have that S/H is a final segment of Γ/H if S is a final segment of Γ.

The proof of the following facts is straightforward:

Lemma

For each subset M of Γ the preimage $\varphi_H^{-1}(M/H)$ of M/H under φ_H is $M + H$. If M is a final segment of Γ, then so is $M + H$.

Preparations for the case of arbitrary rank

Take a final segment S of Γ. We define

$$S^{\diamondsuit}$$

to be the preimage of $\widehat{S/\mathcal{G}(S)}$ under

$$\varphi_{\mathcal{G}(S)} \,:\, \Gamma \,\rightarrow\, \Gamma/\mathcal{G}(S)\,.$$

Note that $S^{\diamondsuit}$ is a final segment of Γ. Observe that $S^{\diamondsuit} = S$ holds if and only if $S/\mathcal{G}(S)$ is closed in $\Gamma/\mathcal{G}(S)$. If $\mathcal{G}(S)$ is trivial, then $S^{\diamondsuit} = \widehat{S}$. Further, $S^{\diamondsuit}$ is closed also if $\mathcal{G}(S)$ is not trivial. We view $S^{\diamondsuit}$ as the "deep closure" of S.

The following gives an interesting interpretation of $S^{\diamondsuit}$.

Proposition

Take a final segment S of Γ. Then $T = S^{\diamondsuit}$ is the largest solution of the equation

$$S + T \,=\, S + S\,.$$

Largest solution of $S_1 + T = S_2$

Theorem

Take final segments S_1 and S_2 of Γ. If condition

$(*)$ $\mathcal{G}(S_1) \subseteq \mathcal{G}(S_2)$, *and* $S_2/\mathcal{G}(S_2)$ *is nonprincipal if* $S_1/\mathcal{G}(S_2)$ *is nonprincipal*

holds, then $T = S_2 -_{\mathrm{fs}} S_1$ *is the largest solution of equation* $S_1 + T = S_2$, *and it is equal to:*

- $S_2 - \alpha$ *if* $S_1/\mathcal{G}(S_2) = (\alpha/\mathcal{G}(S_2))^-$ *is principal,*
- $(S_2 - S_1^c)^\diamondsuit$ *if* $S_1/\mathcal{G}(S_2)$ *is nonprincipal.*

If condition $(*)$ *does not hold, or equivalently, one of the cases*

i) $\mathcal{G}(S_2) \subsetneq \mathcal{G}(S_1)$ *or*

ii) $\mathcal{G}(S_1) = \mathcal{G}(S_2)$ *and* $S_1/\mathcal{G}(S_2)$ *is nonprincipal, but* $S_2/\mathcal{G}(S_2)$ *is principal*

holds, then equation $S_1 + T = S_2$ *has no solution.*

Largest solution of $S_1 + T \subseteq S_2$

For $M \subset \Gamma$, denote by M^+ the largest final segment that is disjoint from M.

Theorem ("shrink final segment")

Take final segments S_1 and S_2 of Γ.

1) If $\mathcal{G}(S_2) \subsetneq \mathcal{G}(S_1)$, then there exists $\alpha \in S_2$ such that

$$S_2' := (\alpha + \mathcal{G}(S_1))^+ \subsetneq S_2 \tag{10}$$

is the largest final segment $S \subset S_2$ having invariance group $\mathcal{G}(S_1)$.

2) Assume that $\mathcal{G}(S_1) = \mathcal{G}(S_2)$ and $S_1/\mathcal{G}(S_1)$ is nonprincipal, but $S_2/\mathcal{G}(S_1)$ is principal, say $S_2/\mathcal{G}(S_1) = (\alpha/\mathcal{G}(S_1))^-$ for some $\alpha \in S_2$. Then S_2' defined as in (10) is the largest final segment contained in S_2 that is nonprincipal.

In both cases, condition $(*)$ of the previous theorem is satisfied with S_2' in place of S_2, and $T = S_2' -_{\rm fs} S_1$ is the largest solution of $S_1 + T = S_2'$. Further, $S_2' -_{\rm fs} S_1 = S_2 -_{\rm fs} S_1$.

The invariance valuation ring of an $\mathcal{O}$-ideal

Take any $\mathcal{O}$-ideal I. We set

$$\mathcal{O}(I) := \{b \in K \mid bI \subseteq I\} \quad \text{and} \quad \mathcal{M}(I) = \{b \in K \mid bI \subsetneq I\}.$$

Then $\mathcal{O}(I)$ is a valuation ring, which we call the invariance valuation ring of I, and $\mathcal{M}(I)$ is its maximal ideal. It turns out that $\mathcal{O}(I)$ is the coarsening (i.e., overring) of $\mathcal{O}$ associated with the invariance group $\mathcal{G}(vI)$ of vI, in the sense that

$$\mathcal{O}(I) = \{b \in K \mid \exists \alpha \in \mathcal{G}(vI) : \alpha \leq vb\}.$$

Denoting the valuation associated with $\mathcal{O}(I)$ by w, we have

$$\forall a \in K : \; wa = va/\mathcal{G}(vI) \quad \text{and} \quad wK = vK/\mathcal{G}(vI).$$

The ring $\mathcal{O}(I)$ is the largest of all valuation rings $\mathcal{O}'$ containing $\mathcal{O}$ such that I is an $\mathcal{O}'$-ideal.

... and its maximal ideal

For $\mathcal{O}$-ideals I, the ideals $\mathcal{M}(I)$ appear in Chapter II, Section 4 of

Laszlo Fuchs – Luigi Salce: *Modules over non-Noetherian domains*, Mathematical Surveys and Monographs **84**, American Mathematical Society, Providence, RI, 2001

under the notation $I^\sharp$. There, properties of $\mathcal{M}(I) = I^\sharp$ are listed. In our paper, we present several of them and show how they follow from our results on final segments in ordered abelian groups.

Largest solution of $I_1 J = I_2$

Theorem

Take $\mathcal{O}$-ideals I_1 and I_2. If condition

$(*)$ $\mathcal{O}(I_1) \subseteq \mathcal{O}(I_2)$, and I_2 is a nonprincipal $\mathcal{O}(I_2)$-ideal if I_1 is a nonprincipal $\mathcal{O}(I_2)$-ideal

holds, then $J = I_2 : I_1$ is the largest solution of the equation $I_1 J = I_2$, and it is equal to:

- $a^{-1} I_2$ *if* $I_1 \mathcal{O}(I_2) = a\mathcal{O}(I_2)$,
- $(I_2(\mathcal{O} : I_1))^{\diamondsuit}$ *if* $I_1 \mathcal{O}(I_2)$ *is a nonprincipal* $\mathcal{O}(I_2)$*-ideal.*

If condition $(*)$ does not hold, or equivalently, one of the cases

i) $\mathcal{O}(I_2) \subsetneq \mathcal{O}(I_1)$ or

ii) $\mathcal{O}(I_1) = \mathcal{O}(I_2)$ and I_1 is a nonprincipal $\mathcal{O}(I_1)$-ideal, but I_2 is a principal $\mathcal{O}(I_1)$-ideal

holds, then the equation $I_1 J = I_2$ has no solution.

Largest solution of $I_1 J \subseteq I_2$

Theorem ("shrink ideal")

Take $\mathcal{O}$-ideals I_1 and I_2.

i) Assume that $\mathcal{O}(I_2) \subsetneq \mathcal{O}(I_1)$. Then there exists $a \in I_2$ such that $I_2' := a\mathcal{M}(I_1) \subsetneq I_2$ is the largest $\mathcal{O}(I_1)$-ideal contained in I_2.

ii) Assume that $\mathcal{O}(I_1) = \mathcal{O}(I_2)$ and I_1 is a nonprincipal $\mathcal{O}(I_1)$-ideal, but I_2 is a principal $\mathcal{O}(I_1)$-ideal, say $I_2 = a\mathcal{O}(I_1)$ for some $a \in K$. In this case, $I_2' := a\mathcal{M}(I_1) \subsetneq I_2$ is the largest nonprincipal $\mathcal{O}(I_1)$-ideal contained in I_2.

In both cases, $J = I_2' : I_1$ is the largest solution of $I_1 J = I_2'$, and $I_2' : I_1 = I_2 : I_1$.

Annihilator of I/IJ

Take $\mathcal{O}$-ideals I_1 and I_2 with $I_2 \subsetneq I_1$. Then the annihilator of the $\mathcal{O}$-module I_1/I_2 is the $\mathcal{O}$-ideal $I_2 : I_1$. Hence, from the previous theorems we obtain:

Theorem

Take $\mathcal{O}$-ideals I and J with $IJ \subsetneq I$. Then the annihilator of the $\mathcal{O}$-module I/IJ is

$$IJ : I = \begin{cases} J\mathcal{O}(IJ) & \text{if } I\mathcal{O}(IJ) \text{ is a principal } \mathcal{O}(IJ)\text{-ideal, and} \\ (J\mathcal{O}(IJ))^{\diamondsuit} & \text{if } I\mathcal{O}(IJ) \text{ is a nonprincipal } \mathcal{O}(IJ)\text{-ideal.} \end{cases}$$

In the special case where $\mathcal{O}(I) = \mathcal{O}(J)$, we have $\mathcal{O}(IJ) = \mathcal{O}(I)$, $I\mathcal{O}(IJ) = I$ and $J\mathcal{O}(IJ) = J$, so that
$IJ : I = J$ if I is a principal $\mathcal{O}(I)$-ideal, and
$IJ : I = J^{\diamondsuit}$ otherwise.

Annihilator of I_1/I_2

Theorem

Take $\mathcal{O}$-ideals I_1, I_2 with $I_2 \subsetneq I_1$.

1) Assume that the equation $I_1 J = I_2$ has a solution.
Then the annihilator of I_1/I_2 is
$a^{-1}I_2$ *if* $I_1\mathcal{O}(I_2) = a\mathcal{O}(I_2)$, *and*
$I_2(\mathcal{O} : I_1)^{\diamondsuit}$ *if* $I_1\mathcal{O}(I_2)$ *is a nonprincipal* $\mathcal{O}(I_2)$*-ideal.*

2) Assume that the equation $I_1 J = I_2$ does not have a solution, and let I_2' be as defined in Theorem "shrink ideal".
Then the annihilator of I_1/I_2 is
$a^{-1}I_2'$ *if* $I_1'\mathcal{O}(I_2) = a\mathcal{O}(I_2)$, *and*
$I_2'(\mathcal{O} : I_1)^{\diamondsuit}$ *if* $I_1\mathcal{O}(I_2)$ *is a nonprincipal* $\mathcal{O}(I_2)$*-ideal.*

Annihilators in special cases

Now we apply our results in order to compute annihilators in special cases. They result from the representation of Kähler differentials in the papers mentioned in the beginning as quotients of ideals of a special form. We consider valued field extensions $(L|K, v)$. We denote by $\mathcal{O}_v$ the valuation ring of (L, v) and by $\mathcal{M}_v$ its maximal ideal.

In particular, we determine when these quotients are annihilated by $\mathcal{M}_v$. This question arose since in the literature it is stated that the annihilators of certain Kähler differentials are almost zero, which means that their annihilator is $\mathcal{M}_v$. However, we found that in several cases of particular interest they are actually zero if they are annihilated by $\mathcal{M}_v$.

Annihilators in special cases

Proposition

Take an $\mathcal{O}_v$-ideal $I \subsetneq \mathcal{O}_v$ and set $J = bI^{n-1}$, where $b \in \mathcal{O}_v$. If $n = 1$, assume that $b \in \mathcal{M}_v$. Then $I/IJ = I/bI^n$ is not zero, and the following statements hold.

1) If there is $a \in K$ such that $vJ/\mathcal{G}(vJ)$ has infimum $va/\mathcal{G}(vJ)$ in $vK/\mathcal{G}(vJ)$ but does not contain this infimum, then

$$\operatorname{ann} I/IJ \;=\; J^{\diamondsuit} \;=\; a\mathcal{O}(J)\,, \tag{11}$$

which properly contains J. In all other cases, $\operatorname{ann} I/IJ = J$.
In all cases, $\mathcal{M}(J) \operatorname{ann} I/IJ \subseteq J$.

2) We have that $\operatorname{ann} I/IJ = \mathcal{M}_v$ if and only if $\mathcal{M}_v$ is principal and either $I = J = \mathcal{M}_v$ (which holds if and only if $n = 2$ and $b \in \mathcal{O}_v^\times$), or $I = \mathcal{O}_v$ and $J = \mathcal{M}_v = (b)$ (which holds if and only if $n = 1$ and $b \in \mathcal{M}_v$).

Annihilators in special cases

Proposition

Take $a \in \mathcal{O}_v$, $n \geq 2$, and an overring $\mathcal{O}$ of $\mathcal{O}_v$ in L with maximal ideal $\mathcal{M}$. Assume that $(a\mathcal{M})^n \neq a\mathcal{M}$. Then the following statements hold.

1) $\operatorname{ann} a\mathcal{M}/(a\mathcal{M})^n =$

$$\begin{cases} (a\mathcal{M})^{n-1} & \text{if } \mathcal{M} \text{ is a principal } \mathcal{O}\text{-ideal,} \\ (a\mathcal{O})^{n-1} = a^{n-1}\mathcal{O} & \text{if } \mathcal{M} \text{ is a nonprincipal } \mathcal{O}\text{-ideal.} \end{cases}$$

2) $\operatorname{ann} a\mathcal{M}/(a\mathcal{M})^n = \mathcal{M}_v$ *if and only if* $n = 2$, $a \notin \mathcal{M}_v$ *and* $\mathcal{M} = \mathcal{M}_v$ *is a principal* $\mathcal{O}_v$*-ideal.*

THE END

Thank you for your attention!

More detailed information

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https://www.fvkuhlmann.de/Fvkslides.html

and a lecture series on valued function fields and the defect can be found on the web page

https://www.fvkuhlmann.de/Fvkls.html.

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