Definable Ranks

Lothar Sebastian Krapp

(joint work-in-progress with Salma Kuhlmann and Lasse Vogel)

Universität Zürich, Institut für Interdisziplinäre Sprachevolutionswissenschaft

06 August 2025

ddg40 : Structures algébriques et ordonnées

Slides are available on:

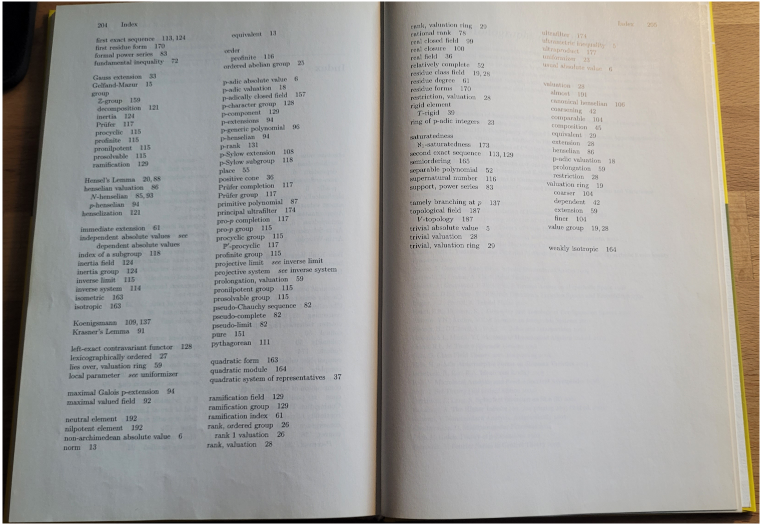
www.krapp-handout.de

1 Ranks

2 Definability

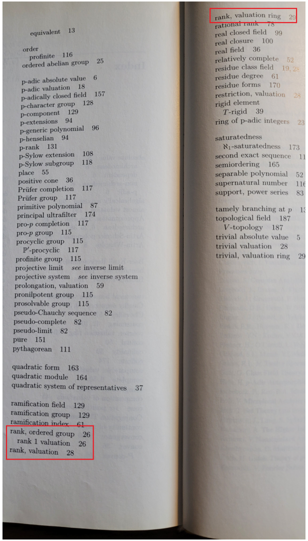
3 Almost real closed fields

Finding the definition


204 Index

Index 205

Finding the definition



Establishing the definition

All orderings are **linear**. Throughout, $K = (K, +, \cdot, 0, 1, <)$ denotes an ordered field, $G = (G, +, 0, <)$ an ordered abelian group and $\Gamma = (\Gamma, <)$ an ordered set.

Definition

We define the respective **ranks** of K, G and Γ as the following sets ordered by set inclusion:

- $\mathrm{rk}_K := \{\mathcal{O} \mid \mathcal{O} \subsetneq K \text{ convex valuation ring of } K\}$
- $\mathrm{rk}_G := \{H \mid H \subsetneq G \text{ convex subgroup of } G\}$,
- $\mathrm{rk}_\Gamma := \{\Delta \mid \Delta \subsetneq \Gamma \text{ final segment of } \Gamma\}$.

Usually the rank is defined as the respective order-type (e.g. "G is of rank 1" if $\mathrm{rk}_G = \{\{0\}\}$). However, we are interested in the actual ordered sets.

Natural valuation on K

Archimedean equivalence classes: for $a \in K$, we set

$$[a] = \{b \in K \mid \exists n \in \mathbb{N}\colon (|a| < n|b| \wedge |b| < n|a|)\}.$$

Natural valuation on K:

$$v_{\mathrm{nat}}\colon K \to \{[a] \mid a \in K\}, a \mapsto [a]$$

defines a valuation where $v_{\mathrm{nat}}(K^{\times})$ is endowed with addition

$$[a] + [b] := [ab], \qquad 0 := [1]$$

and ordering

$$[a] < [b] :\Leftrightarrow ([a] \neq [b] \wedge |a| > |b|).$$

The valuation ring $\mathcal{O}_{v_{\mathrm{nat}}}$ of v_{nat} is the finest convex valuation ring on K. All other convex valuations are coarsenings of v_{nat}.

Natural valuation on G

Archimedean equivalence classes: for $a \in G$, we set

$$[a] = \{b \in G \mid \exists n \in \mathbb{N} \colon (|a| < n|b| \wedge |b| < n|a|)\}.$$

Natural valuation on G:

$$v_G \colon G \to \{[a] \mid a \in G\}, a \mapsto [a]$$

defines a valuation where $v_G(G)$ is endowed with ordering

$$[a] < [b] :\Leftrightarrow ([a] \neq [b] \wedge |a| > |b|).$$

Going down

Starting from an ordered field K, we can "go down" using natural valuations, i.e. naturally associate less complex ordered (algebraic) structures:

$$\begin{array}{rl} & K \\ & \downarrow v_{\text{nat}} \\ v_{\text{nat}}(K^\times) = & G \\ & \downarrow v_G \\ v_G(G \setminus \{0\}) = & \Gamma \end{array}$$

How are the corresponding ranks associated in this setting?

Example: archimedean K

Suppose that K is archimedean (e.g. $K = \mathbb{Q}$ or $K = \mathbb{R}$).

- $\mathrm{rk}_K = \{\mathcal{O} \mid \mathcal{O} \subsetneq K$ convex valuation ring of $K\} = \emptyset$, as the only convex valuation ring of K is the trivial one $\mathcal{O} = K$.
- $\mathrm{rk}_G = \{H \mid H \subsetneq G$ convex subgroup of $G\} = \emptyset$, as $G = \{[1]\} = \{0\}$.
- $\mathrm{rk}_\Gamma = \{\Delta \mid \Delta \subsetneq \Gamma$ final segment of $\Gamma\} = \emptyset$, as $\Gamma = v_G(G \setminus \{0\}) = \emptyset$.

Example: archimedean K

Suppose that K is archimedean (e.g. $K = \mathbb{Q}$ or $K = \mathbb{R}$).

- $\mathrm{rk}_K = \emptyset$
- $\mathrm{rk}_G = \emptyset$
- $\mathrm{rk}_\Gamma = \emptyset$

$\Longrightarrow$ all ranks “equal 0”

Example: archimedean G

Suppose that $G \neq \{0\}$ is archimedean (e.g. $K = k((t))$ the field of formal Laurent series over an ordered field k with $0 < t < \mathbb{Q}^{>0}$ and $G = \mathbb{Z}$).

- $\mathrm{rk}_K = \{\mathcal{O} \mid \mathcal{O} \subsetneq K \text{ convex valuation ring of } K\} = \{\mathcal{O}_{v_{\mathrm{nat}}}\}$, as the only non-trivial convex valuation on K is the natural one.
- $\mathrm{rk}_G = \{H \mid H \subsetneq G \text{ convex subgroup of } G\} = \{\{0\}\}$, as $G \setminus \{0\} = [g]$ for any $g \in G \setminus \{0\}$.
- $\mathrm{rk}_\Gamma = \{\Delta \mid \Delta \subsetneq \Gamma \text{ final segment of } \Gamma\} = \{\emptyset\}$, as $\Gamma = v_G(G \setminus \{0\}) = \{[g]\}$ for any $g \in G \setminus \{0\}$.

Example: archimedean G

Suppose that $G \neq \{0\}$ is archimedean (e.g. $K = k((t))$ the field of formal Laurent series over an ordered field k with $0 < t < \mathbb{Q}^{>0}$ and $G = \mathbb{Z}$).

- $\mathrm{rk}_K = \{\mathcal{O}_{v_{\mathrm{nat}}}\}$
- $\mathrm{rk}_G = \{\{0\}\}$
- $\mathrm{rk}_\Gamma = \{\emptyset\}$

$\Longrightarrow$ all ranks are singletons

Natural order isomorphism

$\mathcal{O}_v = \{a \in K \mid v(a) \geq 0\}$, $\mathcal{I}_v = \{a \in K \mid v(a) > 0\}$.

- For any ordered field K with value group $G = v_{\mathrm{nat}}(K^\times)$,

$$\Phi_K \colon \mathcal{O}_v \mapsto v_{\mathrm{nat}}(\mathcal{O}_v \setminus \mathcal{I}_v)$$

  defines an order preserving bijection from $(\mathrm{rk}_K, \subsetneq)$ to $(\mathrm{rk}_G, \subsetneq)$.

- For any ordered abelian group G with value set $\Gamma = v_G(G \setminus \{0\})$,

$$\Phi_G \colon H \mapsto v_G(H \setminus \{0\})$$

  defines an order preserving bijection from $(\mathrm{rk}_G, \subsetneq)$ to $(\mathrm{rk}_\Gamma, \subsetneq)$.

Natural order isomorphism

$$
\begin{array}{rcl}
K & \qquad & \mathrm{rk}_K \\
\downarrow v_{\mathrm{nat}} & & \wr\!\!\| \downarrow \ \Phi_K \\
v_{\mathrm{nat}}(K^\times) = G & & \mathrm{rk}_G \\
\downarrow v_G & & \wr\!\!\| \downarrow \ \Phi_G \\
v_G(G \setminus \{0\}) = \Gamma & & \mathrm{rk}_\Gamma
\end{array}
$$

All ranks coincide (up to isomorphism).

Model theoretic setting

- first-order languages $\mathcal{L}$:
  $\mathcal{L}_{\mathrm{or}} = \{+, \cdot, 0, 1, <\}$, $\mathcal{L}_{\mathrm{og}} = \{+, 0, <\}$, $\mathcal{L}_{<} = \{<\}$
- $\mathcal{L}$-structures:
  $(k((t)), +, \cdot, 0, 1, <)$, $(\mathbb{Z} \oplus \mathbb{Z}, +, 0, <)$, $(\omega, <)$
- $\mathcal{L}$-formulas and $\mathcal{L}$-sentences:
  $\forall x \exists y\ x + y = 0$
- $\mathcal{L}$-definability (with parameters):
  $\exists y\ y + y = x$ defines $2\mathbb{Z} \oplus 2\mathbb{Z}$ in $(\mathbb{Z} \oplus \mathbb{Z}, +, 0)$ and $k((t))$ in $(k((t)), +, \cdot, 0, 1, <)$.
  $x \cdot x < t$ defines $(-\sqrt{t}, \sqrt{t})$ in $(k((t)), +, \cdot, 0, 1, <)$.

Definable ranks

- $\mathrm{rk}_K := \{\mathcal{O} \mid \mathcal{O} \subsetneq K$ convex valuation ring of $K\}$
- $\mathrm{rk}_G := \{H \mid H \subsetneq G$ convex subgroup of $G\}$,
- $\mathrm{rk}_\Gamma := \{\Delta \mid \Delta \subsetneq \Gamma$ final segment of $\Gamma\}$.

We now consider the subsets of the ranks above consisting of the definable elements.

Definition

We define the respective **definable ranks** of K, G and Γ as the following sets ordered by set inclusion:

- $\mathrm{drk}_K := \{\mathcal{O} \in \mathrm{rk}_K \mid \mathcal{O}$ is $\mathcal{L}_{\mathrm{or}}$-definable in $K\}$
- $\mathrm{drk}_G := \{H \in \mathrm{rk}_G \mid H$ is $\mathcal{L}_{\mathrm{og}}$-definable in $G\}$
- $\mathrm{drk}_\Gamma := \{\Delta \in \mathrm{rk}_\Gamma \mid \Delta$ is $\mathcal{L}_<$-definable in $\Gamma\}$

Remarks

- $\mathrm{drk}_K := \{\mathcal{O} \mid \mathcal{O} \subsetneq K\ \mathcal{L}_{\mathrm{or}}\text{-definable convex valuation ring of } K\}$
- $\mathrm{drk}_G := \{H \mid H \subsetneq G\ \mathcal{L}_{\mathrm{og}}\text{-definable convex subgroup of } G\}$
- $\mathrm{drk}_\Gamma := \{\Delta \mid \Delta \subsetneq \Gamma\ \mathcal{L}_{<}\text{-definable final segment of } \Gamma\}$

Note:

- Examining definable ranks amounts to the study of definable non-trivial convex valuations, definable proper convex subgroups and definable proper final segments, respectively. However, the focus does not lie on the individual definable valuations, subgroups and final segments but on the order-type of the set of *all* such definable substructures.
- Definable ranks for ordered fields, ordered abelian groups and ordered sets can be studied separaterly. However, the connection between the definable ranks is of particular interest in the case $G = v_{\mathrm{nat}}(K^\times)$ and $\Gamma = v_G(G \setminus \{0\})$.

Ansatz

$$
\begin{array}{rcl}
K & & \mathrm{rk}_K \\
\downarrow v_{\mathrm{nat}} & & \wr\!\|\!\downarrow \;\Phi_K \\
v_{\mathrm{nat}}(K^\times) = G & & \mathrm{rk}_G \\
\downarrow v_G & & \wr\!\|\!\downarrow \;\Phi_G \\
v_G(G \setminus \{0\}) = \Gamma & & \mathrm{rk}_\Gamma
\end{array}
$$

How much of the above is still valid if ranks are replaced by definable ranks?

Ansatz

$$\begin{array}{ccc} & K & \mathrm{drk}_K \\ & \downarrow v_{\mathrm{nat}} & ?\downarrow\ \Phi_K \\ v_{\mathrm{nat}}(K^\times) = & G & \mathrm{drk}_G \\ & \downarrow v_G & ?\downarrow\ \Phi_G \\ v_G(G \setminus \{0\}) = & \Gamma & \mathrm{drk}_\Gamma \end{array}$$

How much of the above is still valid if ranks are replaced by definable ranks?

Questions

K ordered field, $G = v_{\mathrm{nat}}(K^\times)$, $\Gamma = v_G(G \setminus \{0\})$.

$$\begin{array}{l} \mathrm{drk}_K \\ \quad ?\downarrow\ \Phi_K\colon \mathcal{O}_v \mapsto v_{\mathrm{nat}}(\mathcal{O}_v \setminus \mathcal{I}_v) \\ \mathrm{drk}_G \\ \quad ?\downarrow\ \Phi_G\colon H \mapsto v_G(H \setminus \{0\}) \\ \mathrm{drk}_\Gamma \end{array}$$

1. What are necessary and sufficient conditions on K such that $\Phi_K(\mathrm{drk}_K) = \mathrm{drk}_G$ and $\Phi_G(\mathrm{drk}_G) = \mathrm{drk}_\Gamma$? – When do Φ_K, Φ_K^{-1}, Φ_G and Φ_G^{-1} preserve definability?
2. Given three ordered sets $(A, <)$, $(B, <)$ and $(C, <)$. When is it possible to construct an ordered field K such that $(\mathrm{drk}_K, \subsetneq) \cong (A, <)$, $(\mathrm{drk}_G, \subsetneq) \cong (B, <)$ and $(\mathrm{drk}_\Gamma, \subsetneq) \cong (C, <)$?

Questions

K ordered field, $G = v_{\mathrm{nat}}(K^\times)$, $\Gamma = v_G(G \setminus \{0\})$.

$$
\begin{array}{l}
\mathrm{drk}_K \\
\quad ?\downarrow\ \Phi_K\colon \mathcal{O}_v \mapsto v_{\mathrm{nat}}(\mathcal{O}_v \setminus \mathcal{I}_v) \\
\mathrm{drk}_G \\
\quad ?\downarrow\ \Phi_G\colon H \mapsto v_G(H \setminus \{0\}) \\
\mathrm{drk}_\Gamma
\end{array}
$$

1. What are necessary and sufficient conditions on K such that $\Phi_K(\mathrm{drk}_K) = \mathrm{drk}_G$ and $\Phi_G(\mathrm{drk}_G) = \mathrm{drk}_\Gamma$?
2. Given three ordered sets $(A,<)$, $(B,<)$ and $(C,<)$. When is it possible to construct an ordered field K such that $(\mathrm{drk}_K, \subsetneq) \cong (A,<)$, $(\mathrm{drk}_G, \subsetneq) \cong (B,<)$ and $(\mathrm{drk}_\Gamma, \subsetneq) \cong (C,<)$?

Example: archimedean K

K archimedean ordered field, $G = v_{\mathrm{nat}}(K^\times)$, $\Gamma = v_G(G \setminus \{0\})$.

Then $\mathrm{rk}_K = \mathrm{rk}_G = \mathrm{rk}_\Gamma = \emptyset$, so $\mathrm{drk}_K = \mathrm{drk}_G = \mathrm{drk}_\Gamma = \emptyset$.

All three definable ranks trivially coincide.

Example: archimedean G

K ordered field, $G = v_{\mathrm{nat}}(K^\times)$ archimedean, $\Gamma = v_G(G \setminus \{0\})$.

Then $\mathrm{rk}_K = \{\mathcal{O}_{v_{\mathrm{nat}}}\}$, $\mathrm{rk}_G = \{\{0\}\}$, $\mathrm{rk}_\Gamma = \{\emptyset\}$.
Thus, $\mathrm{drk}_G = \{\{0\}\}$ and $\mathrm{drk}_\Gamma = \{\emptyset\}$.

In order that all three *definable* ranks coincide, v_{nat} must be definable in K.

- Suppose that G is non-divisible or that the archimedean residue field Kv_{nat} is not real closed. Then v_{nat} is definable [Dittmann, Jahnke, K., Kuhlmann, 2023; Corollary 3.2]. Thus, $\mathrm{drk}_K = \{\mathcal{O}_{v_{\mathrm{nat}}}\}$ and all three definable ranks coincide.
- Let K be the field of formal Puiseux series $K = \bigcup_{n\in\mathbb{N}} \mathbb{R}((t^{1/n}))$ (ordered by $0 < t < \mathbb{Q}^{>0}$). Then K is real closed, whence v_{nat} is not definable due to o-minimality. Thus $\mathrm{drk}_K = \emptyset$, so the definable rank on the field level does not coincide with the definable ranks on value group and value set level.

Ordered fields of generalised power series

Let k be an ordered field and let G be an ordered abelian group. We denote by $k((G))$ the **field of generalised power series**

$$s = \sum_{g \in G} s_g t^g$$

(where $\{g \in G \mid s_g \neq 0\}$ is well-ordered) with coefficients s_g in k and exponents g in G. We denote by $v_{\min}$ the henselian valuation

$$k((G))^{\times} \to G, \quad \sum_{g \geq g_0} s_g t^g \mapsto g_0$$

(for $s_{g_0} \neq 0$).
The field $k((G))$ can be naturally ordered by $\sum_{g \geq g_0} s_g t^g > 0$ if and only if $s_{g_0} > 0$.

Example: G of rank 2

Let $G = \mathbb{Q} \oplus \mathbb{Q}$ be ordered lexicographically, $K = \mathbb{R}((\mathbb{Q} \oplus \mathbb{Q}))$ and $v_G \colon G \to \{1, 2\}$ with $v_G(1, 0) = 1$ and $v_G(0, 1) = 2$.

Then $\mathrm{rk}_G = \{\{0\}, \{0\} \oplus \mathbb{Q}\}$, so $|\mathrm{rk}_K| = |\mathrm{rk}_G| = |\mathrm{rk}_\Gamma| = 2$.

- $\mathrm{drk}_K = \emptyset$, as K is real closed (o-minimal) and therefore the only definable convex valuation ring is the trivial one.
- $\mathrm{drk}_G = \{\{0\}\}$, as G is divisible and therefore the only definable convex subgroup is the trivial one.
- $\mathrm{drk}_\Gamma = \mathrm{rk}_\Gamma = \{\emptyset, \{2\}\}$, as $\Gamma = \{1, 2\}$ is finite and thus any subset of Γ is definable.

Hence, K is a real closed field with $|\mathrm{drk}_K| = 0$, $|\mathrm{drk}_G| = 1$ and $|\mathrm{drk}_\Gamma| = 2$, i.e. with pairwise non-isomorphic definable ranks.

Connecting field and group level

Proposition [K., Kuhlmann, Vogel, 2025]

Let K be an ordered field such that v_{nat} is henselian. Then $\Phi_K(\mathrm{drk}_K) = \mathrm{drk}_G$ if and only if v_{nat} is definable in K.

Questions:

1. When is the natural valuation henselian?
2. When is a henselian natural valuation definable?

Definable valuations

Definition (Delon–Farré, 1996)

A real field F is called **almost real closed**[a] if it admits a henselian valuation v with real closed residue field Fv.

[a] Almost real closed fields are also called **HRC-fields** (Becker, Berr and Gondard, 1999).

- For any ordered abelian group G, the field of generalised power series $\mathbb{R}((G))$ is almost real closed.
- Every almost real closed field can be ordered, but the ordering is not necessarily unique.
- In almost real closed fields, all convex valuations are already henselian. In particular, v_{nat} is the finest henselian valuation (and does thus not depend on the ordering).
- Definable henselian valuations in almost real closed fields were well-studied by Delon–Farré (1996).

Connecting field and group level

Corollary [K., Kuhlmann, Vogel, 2025]

Let K be an an almost real closed field. Then $\Phi_K(\mathrm{drk}_K) = \mathrm{drk}_G$ if and only if v_{nat} is definable in K.

- Delon–Farré (1996) give necessary and sufficient conditions on v_{nat} to be definable.
- Connecting group and set level (i.e. establishing criteria for $\Phi_G(\mathrm{drk}_G) = \mathrm{drk}_\Gamma$) is almost entirely independent of the actual almost real closed field K.

To be continued...

- $\mathrm{drk}_K := \{\mathcal{O} \mid \mathcal{O} \subsetneq K\ \mathcal{L}_{\mathrm{or}}\text{-definable convex valuation ring of } K\}$
- $\mathrm{drk}_G := \{H \mid H \subsetneq G\ \mathcal{L}_{\mathrm{og}}\text{-definable convex subgroup of } G\}$
- $\mathrm{drk}_\Gamma := \{\Delta \mid \Delta \subsetneq \Gamma\ \mathcal{L}_{<}\text{-definable final segment of } \Gamma\}$

Questions:

- Given three ordered sets $(A,<),(B,<),(C,<)$. When is it possible to construct an ordered field K such that $\mathrm{drk}_K \cong (A,<)$, $\mathrm{drk}_G \cong (B,<)$ and $\mathrm{drk}_\Gamma \cong (C,<)$?
- Given an ordered set $(C,<)$. When is it possible to construct an ordered field K such that $\mathrm{drk}_\Gamma \cong (C,<), {\Phi_G}^{-1}(\mathrm{drk}_\Gamma) = \mathrm{drk}_G$ and ${\Phi_K}^{-1}(\mathrm{drk}_G) = \mathrm{drk}_K$, where $G = v_{\mathrm{nat}}(K^\times)$ and $\Gamma = v_G(G \setminus \{0\})$?
- Given an ordered set $(\Gamma,<)$. When is it possible to construct an ordered field K such that $v_G(G \setminus \{0\}) = \Gamma$, $\Phi_K(\mathrm{drk}_K) = \mathrm{drk}_G$ and $\Phi_G(\mathrm{drk}_G) = \mathrm{drk}_\Gamma$, where $G = v_{\mathrm{nat}}(K^\times)$?

References

- E. Becker, R. Berr and D. Gondard, 'Valuation Fans and Residually Real Closed Henselian Fields', *J. Algebra* 215 (1999) 574–602.
- F. Delon and R. Farré, 'Some model theory for almost real closed fields', *J. Symb. Log.* 61 (1996) 1121–1152.
- P. Dittmann, F. Jahnke, L. S. Krapp and S. Kuhlmann, 'Definable valuations on ordered fields', *Model Theory* 2 (2023) 101–120.
- A. J. Engler and A. Prestel, *Valued Fields* (Springer, Berlin, 2005).
- L. S. Krapp, S. Kuhlmann and L. Vogel, 'Definable ranks´, Preprint, arXiv:2506.00443v2.