Fields with the absolute Galois group of $\mathbb{Q}$

K field $\rightsquigarrow$ $G_K := \mathrm{Gal}(\bar K / K)$

Ex.: • $G_K \cong G_{\mathbb{R}} \cong \mathbb{Z}/2 \iff K \equiv \mathbb{R}$ e.g. $K = \mathbb{R} \cap \bar{\mathbb{Q}}$ or $\mathbb{R}((\mathbb{Q}))$

• $G_K \cong G_{\mathbb{Q}_p} \iff K \equiv \mathbb{Q}_p$ e.g. $K = \mathbb{Q}_p^{alg} = \mathbb{Q}_p \cap \bar{\mathbb{Q}}$ or $\mathbb{Q}_p((\mathbb{Q}))$

Conjecture

$G_K \cong G_{\mathbb{Q}} \iff \exists$ hens. v on K with vK div. and $Kv = \mathbb{Q}$

$(\Rightarrow K = \mathbb{Q}$ or $K \equiv \mathbb{Q}((\mathbb{Q})))$

① The local structure

Prop. K any field with $\varphi: G_K \xrightarrow{\cong} G_{\mathbb{Q}}$. Then

(a) char $K = 0$ & $\exists!$ ordering on K

(b) For any $p \in \mathbb{P}$, $\exists!$ p-adic v_p on K with henselization $K_p = \mathrm{Fix}\, \varphi^{-1} G_{\mathbb{Q}_p^{alg}}$

$$\begin{tikzcd}
 & \bar K \arrow[d, dash] & \bar{\mathbb{Q}} \arrow[d, dash] & \\
G_{K_p} \arrow[d, dash] & K_p \arrow[r, phantom, "\cong"] \arrow[d, dash] & \mathbb{Q}_p^{alg} \arrow[d, dash] & G_{\mathbb{Q}_p^{alg}} \arrow[d, dash] \\
\langle G_{K_p} \mid p \in \mathbb{P} \rangle = G_K \arrow[rrr, bend right, "\varphi", "\cong"'] & K & \mathbb{Q} & G_{\mathbb{Q}} = \langle G_{\mathbb{Q}_p^{alg}} \mid p \in \mathbb{P} \rangle
\end{tikzcd}$$

(c) L/K finite Galois

$\Rightarrow$ G_K sees $e_{v_p}(L/K)$ and $f_{v_p}(L/K)$ for all $p \in \mathbb{P}$

& Čebotarev Density Thm.:

e.g. $[L:K] = 2 \Rightarrow \delta(\{p \in \mathbb{P} \mid [LK_p : K_p] = 2\}) = \frac{1}{2}$

Lemma $G_K \cong G_{\mathbb{Q}} \Rightarrow$

(a) for each n, $\#\{L/K \mid [L:K] \le n \;\&\; e_{v_p}(L/K) = 1 \text{ for } p > n\} < \infty$

(b) res: $G_K \to G_{\mathbb{Q}}$ is an isom.

(c) for each n, $\mathbb{Q}^\times/(\mathbb{Q}^\times)^n \xrightarrow{\text{can.}} K^\times/(K^\times)^n$ is an iso.

... Kronecker-Weber, Hasse-Mink., Quadr. Rec., ...

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in memoriam: ALEX PRESTEL

② Reduction to t.d. $K/\mathbb{Q} = 1$

(*) Suppose Conj. true for all L with $G_L \cong G_{\mathbb{Q}}$ & t.d. $L/\mathbb{Q} = 1$

Assume $G_K \cong G_{\mathbb{Q}}$, pick $x \in K \setminus \mathbb{Q}$ and let $L = \overline{\mathbb{Q}(x)} \cap K$

$$\Rightarrow \quad G_K \underset{\cong}{\overset{res}{\twoheadrightarrow}} G_L \underset{\cong}{\overset{res}{\twoheadrightarrow}} G_{\mathbb{Q}}$$

$\overset{(*)}{\Rightarrow}$ $\exists$ hens. w on L with wL div. & $Lw = \mathbb{Q}$

$\Rightarrow$ for v_p on K with coarsening $\tilde{v}_p$ s.t. $\mathcal{O}_{\tilde{v}_p} = \mathcal{O}_{v_p}[\frac{1}{p}]$, $\tilde{v}_p|_L = w$

$\Rightarrow$ $\exists\, 0 \neq y \in \mathfrak{m}_w = \bigcap \mathfrak{m}_{\tilde{v}_p} \cap L$

& $\{0\} \neq \mathfrak{m}_{\tilde{v}_p} = \mathfrak{m}_{\tilde{v}_q}$ for all $p \neq q$ $\Rightarrow$ $\tilde{v}_p = \tilde{v}_q$ non-triv. ✓

③ Proof for $G_K \cong G_{\mathbb{Q}}$ with t.d. $K/\mathbb{Q} = 1$?

Case A: the v_p induce finitely many topologies

$\Rightarrow$ $\exists\, x \in K$ s.t. if $\Sigma^+(x) = \{p \in \mathbb{P} \mid v_p(x) > 0\}$ then $\delta(\Sigma^+(x)) > 0$

$\Rightarrow$ (Čeb. = Prop. (c)): $\Sigma^+(x) = \mathbb{P}$ $\Rightarrow$ ↯

Case B: the v_p induce infinitely many topologies

$\Rightarrow$
- Fund. Thm. of Arith: $x \in K^\times$ uniquely det. by $(v_p(x))_{p \in \mathbb{P} \cup \{\infty\}}$
- $\delta(\Sigma^+(x)) = 0$ for all $x \in K^\times$
- $\bigcap_n (K^\times)^n = \{1\}$
- for each $x \in K^\times$, Lemma (c) gives sequence (x_m) in $\mathbb{Q}$ s.t. (x_m) is p-adic Cauchy for density-1 many p's $\Rightarrow$ almost all v_p induce same top. #

$\Rightarrow$ Case B does not occur $\Rightarrow$

Thm (K. '25) Conj. is true!

Cor. 1 $K \cong \mathbb{Q} \iff G_K \cong G_{\mathbb{Q}}$ & $\forall x, y \in K\, (x^4 + y^4 = 1 \rightarrow xy = 0)$

$\iff G_K \cong G_{\mathbb{Q}}$ & K not large

Cor. 2 **Grothendieck's Birational Section Conj. /$\mathbb{Q}$ is true:**

$\mathcal{C}$ smooth proj. alg. curve /$\mathbb{Q}$

$\Rightarrow$ Any section s of $1 \to G_{\overline{\mathbb{Q}}(\mathcal{C})} \to G_{\mathbb{Q}(\mathcal{C})} \overset{res}{\underset{s}{\rightleftarrows}} G_{\mathbb{Q}} \to 1$ comes from some $P \in \mathcal{C}(\mathbb{Q})$, i.e. $s(G_{\mathbb{Q}}) \leq D_P :=$ dec. of $G_{\mathbb{Q}(\mathcal{C})}$ w.r.t. v_P