Universität
Münster

# Model-Theoretic Tilting Arbitrary Rank Welcome

## DDG 40 : Structures algébriques et ordonnées, Banyuls-sur-Mer

Margarete Ketelsen

4 August 2025

living.knowledge

MM
Mathematics
Münster
Cluster of Excellence

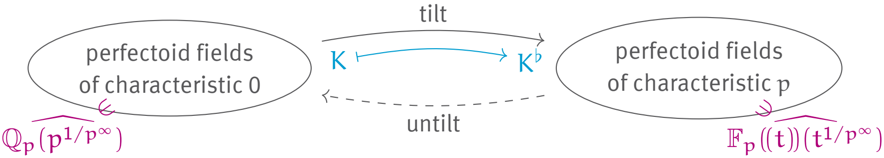
# Introduction

## Perfectoid fields & Tilting

### Theorem (Fontaine–Wintenberger ’79)

$$\mathrm{Gal}(\mathbb{Q}_p(p^{1/p^\infty})) \simeq \mathrm{Gal}(\mathbb{F}_p((t))(t^{1/p^\infty})).$$

GENERALIZATION [SCHOLZE ’12]: framework of perfectoid fields and tilting



**Perfectoid transfer:** $K$ and $K^\flat$ are very similar.

⇝ Can also be explained with model theory, [RIDEAU-KIKUCHI–SCANLON–SIMON ’25], [JAHNKE–KARTAS ’25].

# Valued Fields

## Notation

A **valued field** $(K, v)$ is a field $K$ together with a valuation map

$$v \colon K^{\times} \twoheadrightarrow \Gamma.$$

ordered abelian group,
written additively

$\rightsquigarrow$ related algebraic objects:

- $vK \coloneqq \Gamma$, the **value group**,
- $\mathcal{O}_v \coloneqq \{x \in K^{\times} : v(x) \geqslant 0\}$, the **valuation ring** with maximal ideal $\mathfrak{m}_v$, and
- $Kv \coloneqq \mathcal{O}_v/\mathfrak{m}_v$, the **residue field**, with the **residue map** $\mathrm{res}_v \colon \mathcal{O}_v \twoheadrightarrow Kv$.

THE CHARACTERISTIC
OF A VALUED FIELD:

$(\mathrm{char}(K), \mathrm{char}(Kv)) =$

$\to (0, 0)$, **equicharacteristic zero**

$\to (0, p)$, **mixed characteristic**

$\to (p, p)$, **equicharacteristic $p$** / **positive characteristic**

# Perfectoid Fields

A **perfectoid field** is a valued field $(K, v)$ of residue characteristic $\mathrm{char}(Kv) = p > 0$ such that

tilting is possible

**(1)** complete and the value group has rank 1,

**(2)** $vK$ is $p$-divisible, and

**(3)** $\mathcal{O}_v/p$ is **semiperfect**,
i.e., the Frobenius $\mathcal{O}_v/p \to \mathcal{O}_v/p, x \mapsto x^p$ is surjective.

the tilt is nice

**We want to remove (1) and still be able to define the tilt.**

# A New Class of Valued Fields

Consider the class $\mathcal{C}_p$ of **henselian semitame fields of mixed characteristic** $(0, p)$, i.e., valued fields $(K, v)$ of mixed characteristic $(0, p)$ where

**(1)*** $(K, v)$ is henselian,   NEW!

**(2)** $vK$ is $p$-divisible, and

**(3)** $\mathcal{O}_v/p$ is semiperfect.

Let $\mathcal{C} := \bigcup_{p \text{ prime}} \mathcal{C}_p$ be the class of **henselian semitame fields of mixed characteristic**.

FACT:

- Perfectoid fields of mixed characteristic are contained in $\mathcal{C}$
- $\mathcal{C}_p$ is an elementary class of valued fields,
- $\mathcal{C}$ is closed under elementary equivalence.

AIM FOR TODAY: Define a **model-theoretic tilt** for valued fields in $\mathcal{C}$.

margarete.ketelsen@uni-muenster.de

# Main Tool: The Standard Decomposition

## Understanding valued fields of higher rank

Let $(K, v)$ be a valued field of mixed characteristic. (Think of the arrows as *places*/residue map)
We can decompose into:

$$\begin{tikzcd} K \arrow[r, "v_0"] & K_0 \arrow[r, "\overline{v_p}"] & K_p \arrow[r, "\overline{\overline{v}}"] & Kv \end{tikzcd}$$

equi 0

mixed
rank 1

equi $p$

**Fact:** If $(K, v)$ is $\aleph_1$-saturated, then $(K_0, \overline{v_p})$ is complete. ⇢ We can tilt!

# The Model-Theoretic Tilt

## Definition

Let $(K, v) \in \mathcal{C}$ and take $(L, w) \models \mathrm{Th}(K, v)$ to be $\aleph_1$-saturated. We construct the **tilt** $(L^\flat, w^\flat)$ of $(L, w)$ as follows:

$$\begin{tikzcd}
L \arrow[r, "w_0", "\Delta := w_0 L"'] & L_0 \arrow[r, "\overline{w_p}", "\text{perfectoid}"'] \arrow[d, squiggly, "\text{tilt}"'] & L_p \arrow[r, "\overline{\overline{w}}"] & Lw \\
L^\flat := (L_0)^\flat((\Delta)) \arrow[r, "v_\Delta", "\text{Hahn series}"'] \arrow[rrr, bend right, "w^\flat"'] & (L_0)^\flat \arrow[r, "\overline{w_p}^\flat"] & L_p \arrow[r, "\overline{\overline{w}}"] & Lw
\end{tikzcd}$$

**Theorem (K.):** $\mathrm{Th}(L^\flat, w^\flat)$ does not depend on the choice of the saturated model $(L, w)$.

**Definition:** The **tilt** of $\mathrm{Th}(K, v)$ is $\boxed{\mathrm{Th}^\flat(K, v) := \mathrm{Th}(L^\flat, w^\flat).}$

# The Model-Theoretic Tilt

## Well-definedness

### Theorem (K.)

$\mathrm{Th}(L^\flat, w^\flat)$ *does not depend on the choice of the saturated model* $(L, w)$.

PROOF INGREDIENTS:

**(1)** FACT [GITIN]: If $(L, w) \equiv (L', w')$ are both $\aleph_0$-saturated, then

- $(L, w_0) \equiv (L', w_0')$
- $(L_0, \overline{w_p}) \equiv (L_0', \overline{w_p'})$
- $(L_p, \overline{\overline{w}}) \equiv (L_p', \overline{\overline{w'}})$

**(2)** FACT [JAHNKE–KARTAS '25]: If $(K, v) \equiv (K', v')$ are both **perfectoid**, then $(K^\flat, v^\flat) \equiv (K'^\flat, v'^\flat)$.

**(3)** $\mathrm{AKE}^{\equiv}$ FOR EQUICHARACTERISTIC TAME FIELDS [KUHLMANN '16] for the Hahn series.

**(4)** DECOMPOSITION AKE [K.]: We can glue everything back together. (follows from [KUHLMANN '16])

margarete.ketelsen@uni-muenster.de

# The Model-Theoretic Tilt of a Perfectoid Field

## Theorem (K.)

*Let* $(K, v)$ *be a perfectoid field. Then* $\mathrm{Th}^\flat(K, v) = \mathrm{Th}(K^\flat, v^\flat)$.

PROOF SKETCH: Let $(L, w) := (K, v)^{\mathcal{U}}$. Need to show: $(L^\flat, w^\flat) \equiv (K^\flat, v^\flat)$.

$$\begin{tikzcd}
 & L \arrow[r, "w_0", "\Delta := w_0 L"'] & L_0 \arrow[r, "\overline{w_p}", "\text{perfectoid}"', ""{name=T}] & L_p \arrow[r, "\overline{\overline{w}}"] & Lw \\
 & & & & \ell \\
L^\flat := (L_0)^\flat((\Delta)) \arrow[r, "v_\Delta", "\text{Hahn series}"'] & (L_0)^\flat \arrow[r, "\overline{w_p}^\flat", ""'{name=S}] & L_p \arrow[rr, "\overline{\overline{w}}", ""'{name=A}] \arrow[urr, dashed, "\text{tame}", "\equiv \mathbb{Q}"'] & & Lw \\
 & & K^\flat \arrow[rr, "v^\flat", ""{name=B}] & & K^\flat v^\flat
\arrow[from=T, to=S, rightsquigarrow, "\text{tilt}"']
\arrow[from=B, to=A, hook, "\preceq", "\text{[Jahnke–Kartas '25]}"']
\end{tikzcd}$$

$\underbrace{L^\flat \xrightarrow{v_\Delta} (L_0)^\flat \xrightarrow{\overline{w_p}^\flat} L_p}_{\text{tame with divisible value group}}$

$$\overset{\text{tame AKE}^{\equiv}}{\Longrightarrow} (K^\flat, v^\flat) \preceq (L_p, \overline{\overline{w}}) \equiv (L^\flat, w^\flat)$$

[KUHLMANN '16]

margarete.ketelsen@uni-muenster.de

# Fontaine–Wintenberger theorem

## Theorem (Scholze, Fontaine–Wintenberger for perfectoid fields)

*Let* $(K, v)$ *be a perfectoid field. Then* $\mathrm{Gal}(K) \cong \mathrm{Gal}(K^\flat)$.

## Theorem (K., Fontaine–Wintenberger for the model-theoretic tilt)

*Let* $(L, w) \in \mathcal{C}$ *be* $\aleph_1$*-saturated. Then* $\mathrm{Gal}(L) \cong \mathrm{Gal}(L^\flat)$.

PROOF SKETCH:

$$\begin{tikzcd}
L \arrow[r, "w_0", "\Delta := w_0 L"'] & L_0 \arrow[r, "\overline{w_p}", "\text{perfectoid}"'] & L_p \arrow[r, "\overline{\overline{w}}"] & Lw \\
\mathrm{Gal}(L) & \mathrm{Gal}(L_0) & {} \arrow[dd, rightsquigarrow, "\text{tilt}"'] & \\
\wr\| & \wr\| \arrow[l, Rightarrow, "\text{ramification}", "\text{theory}"'] & & \\
\mathrm{Gal}(L^\flat) & \mathrm{Gal}(L_0^\flat) & {} & \\
L^\flat := (L_0)^\flat((\Delta)) \arrow[r, "v_\Delta", "\text{Hahn series}"'] & (L_0)^\flat \arrow[r, "\overline{w_p}^\flat"] & L_p \arrow[r, "\overline{\overline{w}}"] & Lw
\end{tikzcd}$$

perfectoid tilt

# $C_i$-transfer

## Definition

A field $K$ is $C_i$ if every homogeneous polynomial over $K$ of degree $d$ in $n > d^i$ many variables has a non-trivial zero in $K$.

FACT [JAHNKE–KARTAS '25]: Let $(K, v)$ be a perfectoid field. If $K$ is $C_i$, then so is $K^\flat$.

## Theorem (K.)

*Let* $(L, w) \in \mathcal{C}$ *be* $\aleph_1$*-saturated. If* $L$ *is* $C_i$*, then so is* $L^\flat$.

PROOF SKETCH:

$$\begin{tikzcd}
L \arrow[r, "w_0", "\Delta := w_0 L"'] & L_0 \arrow[r, "\overline{w_p}", "\text{perfectoid}"'] \arrow[ddd, rightsquigarrow, "\text{tilt}"', shift left=8] & L_p \arrow[r, "\overline{\overline{w}}"] & Lw \\
C_i \arrow[r, Rightarrow, "(*)"'] & C_{i-n} \arrow[d, Rightarrow] & & \\
C_i & C_{i-n} \arrow[l, Rightarrow, "\text{Greenberg}"'] & & \\
L^\flat := (L_0)^\flat((\Delta)) \arrow[r, "v_\Delta", "\text{Hahn series}"'] & (L_0)^\flat \arrow[r, "\overline{w_p}^\flat"] & L_p \arrow[r, "\overline{\overline{w}}"] & Lw
\end{tikzcd}$$

perfectoid tilt

$(*)$: $K((t))$ is $C_i \implies K$ is $C_{i-1}$

(Greenberg): $K$ is $C_i \implies K((t))$ is $C_{i+1}$

# Summary

- We defined a **class $\mathcal{C}$** of henselian semitame fields of mixed characteristic. This class contains all perfectoid fields of mixed characteristic, but also valued fields with arbitrary rank.
- We defined a **model-theoretic tilt** for valued fields in $\mathcal{C}$. It is defined **up to elementary equivalence**. MAIN TOOL: the standard decomposition.
- We proved that the model-theoretic tilt **of a perfectoid field** is the same as its usual tilt (up to elementary equivalence).
- We showed a version of **Fontaine-Wintenberger** for the model-theoretic tilt.
- We proved transfer of the **$C_i$-property**.

**Thank you!**

margarete.ketelsen@uni-muenster.de

# References

- Jean-Marc Fontaine and Jean-Pierre Wintenberger, **Le “corps des normes” de certaines extensions algébriques de corps locaux**, C.R. Acad. Sci. Paris Sér. AB **288.6** (1979), A367-A370.
- Leo Gitin, **Model theory for commutative algebra with applications to valued fields**, upcoming work.
- Franziska Jahnke and Konstantinos Kartas, **Beyond the Fontaine-Wintenberger theorem**, to appear in J. Amer. Math. Soc. (Published online 2025), `doi:10.1090/jams/1056`.
- Margarete Ketelsen, **Decomposition AKE for tame fields**, upcoming work.
- Margarete Ketelsen, **Model-theoretic tilting – Arbitrary rank welcome**, upcoming work.
- Silvain Rideau-Kikuchi, Thomas Scanlon and Pierre Simon, **The tilting equivalence as a bi-interpretation**, arXiv preprint (2025), `arXiv:2505.01321`.
- Franz-Viktor Kuhlmann, **The algebra and model theory of tame valued fields**, J. Reine Angew. Math. **2016** (2016), no. 719, 1–43.
- Franz-Viktor Kuhlmann and Anna Rzepka, **The valuation theory of deeply ramified fields and its connection with defect extensions**, Trans. Amer. Math. Soc. **376** (2023), no. 4, 2693–2738.
- Peter Scholze, **Perfectoid spaces**, Publ. math. IHÉS **116** (2012), no. 1, 245–313.

margarete.ketelsen@uni-muenster.de