A non-model-complete pfaffian chain

Joint work with Siegfried van Hille, Jonathan Kirby, and Patrick Speissegger

O-minimal structures

We write $\bar{\mathbb{R}} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$ for the ordered field of real numbers.

O-minimal structures

We write $\bar{\mathbb{R}} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$ for the ordered field of real numbers. An expansion $\tilde{\mathbb{R}}$ of $\bar{\mathbb{R}}$ is said to be o-minimal if every subset of $\mathbb{R}$ definable in $\tilde{\mathbb{R}}$ is a finite union of points and open intervals.

O-minimal structures

We write $\bar{\mathbb{R}} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$ for the ordered field of real numbers. An expansion $\tilde{\mathbb{R}}$ of $\bar{\mathbb{R}}$ is said to be o-minimal if every subset of $\mathbb{R}$ definable in $\bar{\mathbb{R}}$ is a finite union of points and open intervals.

Examples:

- $\bar{\mathbb{R}}$ is o-minimal (follows from Tarski's quantifier elimination).

O-minimal structures

We write $\bar{\mathbb{R}} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$ for the ordered field of real numbers. An expansion $\tilde{\mathbb{R}}$ of $\bar{\mathbb{R}}$ is said to be o-minimal if every subset of $\mathbb{R}$ definable in $\tilde{\mathbb{R}}$ is a finite union of points and open intervals.

Examples:

- $\bar{\mathbb{R}}$ is o-minimal (follows from Tarski's quantifier elimination).
- $\mathbb{R}_{\mathrm{exp}} = (\bar{\mathbb{R}}, \exp)$ is o-minimal (Wilkie).

Pfaffian functions

Khovanskii introduced Pfaffian functions in the 1980s, and his theory was very influential in the development of o-minimal structures.

Pfaffian functions

Khovanskii introduced Pfaffian functions in the 1980s, and his theory was very influential in the development of o-minimal structures.

A sequence $f_1, \ldots, f_l : (a, b) \to \mathbb{R}$ of analytic functions is a pfaffian chain if there are polynomials $p_1, \ldots, p_l$ such that

$$f_i'(t) = p_i(t, f_1(t), \ldots, f_i(t))$$

for t in (a, b) and $i = 1, \ldots, l$.

Pfaffian functions

Khovanskii introduced Pfaffian functions in the 1980s, and his theory was very influential in the development of o-minimal structures.
A sequence $f_1, \ldots, f_l : (a, b) \to \mathbb{R}$ of analytic functions is a pfaffian chain if there are polynomials $p_1, \ldots, p_l$ such that

$$f_i'(t) = p_i(t, f_1(t), \ldots, f_i(t))$$

for t in (a, b) and $i = 1, \ldots, l$.
Wilkie showed that if $f_1, \ldots, f_l$ is a pfaffian chain then $(\bar{\mathbb{R}}, f_1, \ldots, f_l)$ is o-minimal.

Model completeness

A theory T is model complete if every formula is equivalent to an existential formula.

Model completeness

A theory T is model complete if every formula is equivalent to an existential formula.

To prove o-minimality of $\mathbb{R}_{\exp}$, Wilkie showed that its theory is model complete. In the same paper, Wilkie showed that if $f_1, \ldots, f_l : [0,1] \to \mathbb{R}$ is a pfaffian chain (analytic at the endpoints) then the theory of $(\bar{\mathbb{R}}, f_1, \ldots, f_l)$ is model complete.

Model completeness

A theory T is model complete if every formula is equivalent to an existential formula.
To prove o-minimality of $\mathbb{R}_{\exp}$, Wilkie showed that its theory is model complete. In the same paper, Wilkie showed that if $f_1, \ldots, f_l : [0,1] \to \mathbb{R}$ is a pfaffian chain (analytic at the endpoints) then the theory of $(\bar{\mathbb{R}}, f_1, \ldots, f_l)$ is model complete.
Wilkie's o-minimality proof for unrestricted pfaffian functions didn't go via model completeness, and the question of model completeness for unrestricted pfaffian chains remained open.

Main result

We show the following (joint work with van Hille, Kirby and Speissegger).

Theorem

There is a pfaffian chain $f_1, \ldots, f_l : (0, 1) \to \mathbb{R}$ such that the theory of $(\bar{\mathbb{R}}, f_1, \ldots, f_l)$ is not model complete.

The j-function

The j-function is a classical modular function. It is holomorphic on the upper half-plane, invariant under $SL_2(\mathbb{Z})$, and is real valued on the imaginary axis. Peterzil and Starchenko showed that, restricted to its standard fundamental domain, j is definable in an o-minimal structure.

The j-function

The j-function is a classical modular function. It is holomorphic on the upper half-plane, invariant under $SL_2(\mathbb{Z})$, and is real valued on the imaginary axis. Peterzil and Starchenko showed that, restricted to its standard fundamental domain, j is definable in an o-minimal structure.

To work with a real function, we put $f(t) = j(it)$ for $t > 1$. We show

Theorem

The theory of the structure $(\bar{\mathbb{R}}, f, f', f'', f''', \ldots)$ is not model complete.

The j-function

The j-function is a classical modular function. It is holomorphic on the upper half-plane, invariant under $SL_2(\mathbb{Z})$, and is real valued on the imaginary axis. Peterzil and Starchenko showed that, restricted to its standard fundamental domain, j is definable in an o-minimal structure.

To work with a real function, we put $f(t) = j(it)$ for $t > 1$. We show

Theorem

The theory of the structure $(\bar{\mathbb{R}}, f, f', f'', f''', \ldots)$ is not model complete.

Note that j satisfies a third order differential equation, so the other derivatives are rational in the first three.

Outline of proof

We use an idea originally due to Bianconi (in a different context).

Outline of proof

We use an idea originally due to Bianconi (in a different context). Suppose that the theory of $(\bar{\mathbb{R}}, f, f', f'', f''', \ldots)$ is model complete.

Outline of proof

We use an idea originally due to Bianconi (in a different context). Suppose that the theory of $(\bar{\mathbb{R}}, f, f', f'', f''', \ldots)$ is model complete.

- The j function grows exponentially as we go up the imaginary axis. So the structure $(\bar{\mathbb{R}}, f, f', f'', f''', \ldots)$ has a definable function of non-polynomial growth. By an amazing theorem due to Chris Miller, the exponential function on $\mathbb{R}$ is definable in this structure.

Outline of proof

We use an idea originally due to Bianconi (in a different context). Suppose that the theory of $(\bar{\mathbb{R}}, f, f', f'', f''', \ldots)$ is model complete.

- The j function grows exponentially as we go up the imaginary axis. So the structure $(\bar{\mathbb{R}}, f, f', f'', f''', \ldots)$ has a definable function of non-polynomial growth. By an amazing theorem due to Chris Miller, the exponential function on $\mathbb{R}$ is definable in this structure.
- By model completeness, exp is existentially definable.

Outline of proof, continued

- From a result by Wilkie and me (which builds on some of the tools in Wilkie's model completeness proof for $\mathbb{R}_{\exp}$) there are analytic functions $\phi_2, \ldots, \phi_n : (a, \infty) \to \mathbb{R}$ such that

$$\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}\Big(t, \phi_1(t), \ldots, \phi_n(t), f(t), f(\phi_1(t)), \ldots, f(\phi_n(t)),$$

$$f'(t), f'(\phi_1(t)), \ldots, f'(\phi_n(t)), f''(t), f''(\phi_1(t)), \ldots, f''(\phi_n(t))\Big)$$

$$\leq 3n + 4$$

where $\phi_1 = \exp$.

Finishing the proof

Finishing the proof

We use the following result, due to Blázquez-Sanz, Casale, Freitag and Nagloo:

Theorem

If $\psi_1, \ldots, \psi_m$ are germs of analytic functions at 0 in $\mathbb{C}$ taking values in the upper half plane and are suitably independent, then

$$\mathrm{trdeg}_{\mathbb{C}}\mathbb{C}\Big(t, \psi_1(t), \ldots, \psi_m(t), j(t+i), j(\psi_1(t)), \ldots, j(\psi_m(t)),$$
$$j'(t+i), j'(\psi_1(t)), \ldots, j'(\psi_m(t)), j''(t+i), j''(\psi(t)), \ldots, j''(\psi_m(t)), \exp(t)\Big)$$
$$\geq 3m+5$$

Finishing the proof

We use the following result, due to Blázquez-Sanz, Casale, Freitag and Nagloo:

Theorem

If $\psi_1, \ldots, \psi_m$ are germs of analytic functions at 0 in $\mathbb{C}$ taking values in the upper half plane and are suitably independent, then

$$\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}\Big(t, \psi_1(t), \ldots, \psi_m(t), j(t+i), j(\psi_1(t)), \ldots, j(\psi_m(t)),$$
$$j'(t+i), j'(\psi_1(t)), \ldots, j'(\psi_m(t)), j''(t+i), j''(\psi(t)), \ldots, j''(\psi_m(t)), \exp(t)\Big)$$
$$\geq 3m+5$$

In our situation, we can show that we can assume the independence condition, and translating to f we get a contradiction. Hence the theory of $(\bar{\mathbb{R}}, f, f', f'', f''', \ldots)$ is not model complete.

Some bad news

Some bad news

Freitag showed

Theorem

The j-function is not pfaffian.

Switching to the inverse

We can get round the bad news by using the fact that there is a pfaffian function $g : (0,1) \to \mathbb{R}$ such that

$$j\left(\frac{ig(1-z)}{g(z)}\right) = 256\frac{(z^2-z+1)^3}{z^2(1-z)^2}.$$

Switching to the inverse

We can get round the bad news by using the fact that there is a pfaffian function $g : (0,1) \to \mathbb{R}$ such that

$$j\left(\frac{ig(1-z)}{g(z)}\right) = 256\frac{(z^2-z+1)^3}{z^2(1-z)^2}.$$

Here g is a classical hypergeometric function:

$$g(z) = \sum_{m=0}^{\infty} \frac{(2m)!^2}{2^{4m}m!^4} z^m$$

Switching to the inverse

We can get round the bad news by using the fact that there is a pfaffian function $g : (0,1) \to \mathbb{R}$ such that

$$j\left(\frac{ig(1-z)}{g(z)}\right) = 256\frac{(z^2-z+1)^3}{z^2(1-z)^2}.$$

Here g is a classical hypergeometric function:

$$g(z) = \sum_{m=0}^{\infty} \frac{(2m)!^2}{2^{4m}m!^4} z^m$$

Using this and the theorem, we can show that the pfaffian chain for g is not model complete:

Theorem

The theory of $(\bar{\mathbb{R}}, 1/z(z-1), g'/g, g)$ is not model complete.

Some open questions

It can be shown that exp is the only obstruction to model completeness of $(\bar{\mathbb{R}}, 1/z(z-1), g'/g, g)$, in that the theory of $(\bar{\mathbb{R}}, 1/z(z-1), g'/g, g, \exp)$ is model complete.

Some open questions

It can be shown that exp is the only obstruction to model completeness of $(\bar{\mathbb{R}}, 1/z(z-1), g'/g, g)$, in that the theory of $(\bar{\mathbb{R}}, 1/z(z-1), g'/g, g, \exp)$ is model complete.
So a natural question is:

Question

Suppose $f_1, \ldots, f_l : (a, b) \to \mathbb{R}$ is a pfaffian chain. Is the theory of $(\bar{\mathbb{R}}, f_1, \ldots, f_l, \exp)$ model complete?

Some open questions

It can be shown that exp is the only obstruction to model completeness of $(\bar{\mathbb{R}}, 1/z(z-1), g'/g, g)$, in that the theory of $(\bar{\mathbb{R}}, 1/z(z-1), g'/g, g, \exp)$ is model complete.
So a natural question is:

Question

Suppose $f_1, \ldots, f_l : (a, b) \to \mathbb{R}$ is a pfaffian chain. Is the theory of $(\bar{\mathbb{R}}, f_1, \ldots, f_l, \exp)$ model complete?

I think here the answer is no, but I haven't checked any details.

Some open questions

It can be shown that exp is the only obstruction to model completeness of $(\bar{\mathbb{R}}, 1/z(z-1), g'/g, g)$, in that the theory of $(\bar{\mathbb{R}}, 1/z(z-1), g'/g, g, \exp)$ is model complete.
So a natural question is:

Question

Suppose $f_1, \ldots, f_l : (a, b) \to \mathbb{R}$ is a pfaffian chain. Is the theory of $(\bar{\mathbb{R}}, f_1, \ldots, f_l, \exp)$ model complete?

I think here the answer is no, but I haven't checked any details.

Question

Is the theory of the expansion $\mathbb{R}_{\text{Pfaff}}$ of the real field by all pfaffian functions (on intervals) model complete?

Thank you!