Preface

Themes of the talk chosen in Prestel's memory, who was my "academic grandfather", and whose ideas have shaped my approach to mathematics. I met him first during my undergraduate studies in Freiburg, and many times since, in Konstanz and most notably in Oberwolfach in January 2016, where we had several discussions around definability of henselian valuations.

§1 t-henselianity

Throughout the talk: K field
We consider field topologies on K, given by a neighbourhood base τ of 0. $\rightsquigarrow (K,\tau)$

Def.: A V-topology on K is a non-discrete Hausdorff field topology on K s.th.
$\forall U \in \tau \ \exists W \in \tau \quad x \cdot y \in W \to x \in U \vee y \in U$
"Product is only small if one of the factors is"

Example: Topologies induced on a field by
- an ordering or
- a valuation

are V-topologies.

Fact (Kowalsky-Dürbaum): (K,τ) topological field.
Then τ is a V-topology iff there exists an archimedean absolute value or a valuation on K whose induced topology coincides with τ.

Def: A V-topological field (K,τ) is t-henselian if for every $n \geq 1$ ex. $U \in \tau$ s.th. every polynomial
$f = X^{n+1} + X^n + \sum_{i=0}^{n-1} u_i X^i$ with $u_i \in U$
has a root in K.

Recall: (K,v) is henselian iff every polynomial
$f = X^{n+1} + X^n + \sum_{i=0}^{n-1} m_i X^i$ with $m_i \in \mathfrak{m}_v$
has a root in K.

Fact (Prestel–Ziegler '78)

(A) If (K,τ) is t-henselian and $\aleph_0$-saturated, then there is a hens. valuation on K inducing τ.

(B) There are t-henselian fields that are neither sep. closed nor real closed which admit no non-trivial henselian valuation.

Fact (Prestel '91):

Let (K,v) be henselian and assume $g \in K[X]$ irred. and separable with $\deg(g) > 1$.
Take $a \in K$ with $g'(a) \neq 0$. Then, the sets

$$\left(c \cdot \left\{ \frac{1}{g(x)} - \frac{1}{g(a)} : x \in K \right\} \right)_{c \in K^\times}$$

give a neighbourhood base of τ_v.

Corollary: (K,v) (non-trivial) henselian, K not sep. closed. Then any $L \equiv_{\mathcal{L}_{ring}} K$ admits a t-henselian V-topology.

§2 Properties of henselian valuations preserved by $\mathcal{L}_{ring}$-elementary equivalence

Observation: (K,v) henselian, $\operatorname{char}(K) = 0 \neq \operatorname{char}(Kv) = p$.
Then any $L \equiv_{\mathcal{L}_{ring}} K$ admits a henselian valuation of mixed characteristic.

Pf: By Keisler–Shelah, there is an ultrafilter $\mathcal{U}$ with $K^{\mathcal{U}} \cong L^{\mathcal{U}}$ as fields.
Consider $v^{\mathcal{U}}$ on $K^{\mathcal{U}}$ $\Rightarrow$ $v^{\mathcal{U}}\restriction_L$ is henselian and has mixed characteristic. $\boxtimes$

Def: A valued field (K,v) of mixed char $(0,p)$ is **perfectoid** if
(i) (K,v) is complete and $vK \leq \mathbb{R}$ and
(ii) $v(p)$ is not minimum positive and
(iii) $\mathcal{O}_v/(p)$ is semiperfect, i.e. $x \mapsto x^p$ is surj.

Theorem (J.-Lavaud, WIP): The class
$\mathcal{C}_{0,p} = \{ K$ field : $\mathrm{char}(K) = 0$ and K admits a henselian valuation v of char $(0,p)$ s.th. vK is regular, $v(p)$ is not min. pos. and $\mathcal{O}_v/(p)$ is semiperfect $\}$
is the $\mathcal{L}_{ring}$-elementary class generated by all perfectoid fields of char $(0,p)$.
In fact, for any $K \in \mathcal{C}_{0,p}$ there is a perfectoid field L with $K \equiv_{\mathcal{L}_{ring}} L$.

Proof steps:
(0.) If (K,v) is perfectoid of char $(0,p)$, then $(K,v) \in \mathcal{C}_{0,p}$. ✓
(1.) $\mathcal{C}_{0,p}$ is closed under ultraproducts. ✓
(2.) $\mathcal{C}_{0,p}$ is closed under $\mathcal{L}_{ring}$-elem. equivalence:
(3.) For any $K \in \mathcal{C}_{0,p}$ there is a perfectoid (L,w) with $K \equiv_{\mathcal{L}_{ring}} L$.

Proof sketch of (2.): Take $K \in \mathcal{C}_{0,p}$, v hens. on K witnessing $K \in \mathcal{C}_{0,p}$, $L \equiv_{\mathcal{L}_{ring}} K$

Obs.
$\Rightarrow$ L admits a henselian valuation w of mixed char, wlog it has no mixed char coarsenings.

We show: w witnesses that $L \in \mathcal{C}_{0,p}$.

Standard decomposition of (L, w):

$$L \xrightarrow{(0,0)} Lw_0 \xrightarrow{\text{rk } 1} Lw \qquad \text{with } \mathcal{O}_{w_0} = \mathcal{O}_w[1/p]$$

We need to show:

- wL is regular, i.e. $w_0 L \equiv \mathbb{Q}$
- $w(p)$ is not min. pos., i.e. $\langle w(p) \rangle \not\equiv \mathbb{Z}$
- $\mathcal{O}_w/p$ is semiperfect, i.e. for some (equiv: all) $(L^*, w^*) \succ (L, w)$ $\aleph_1$-sat., all coarsenings of w^* have perfect residue field.

Keisler-Shelah: Let $F \cong K^u \cong L^u$ be a common ultrapower, w^u and v^u with
$(F, w^u) \succ (L, w)$ $\qquad (F, v^u) \succ (K, v)$.
Wlog (F, w^u), (F, v^u) $\aleph_1$-sat.

Standard decomposition of (F, v^u):

$$F \xrightarrow[\text{div.}]{(0,0)} F(v^u)_0 \xrightarrow[\cong \mathbb{R}]{(0,p) \text{ rk } 1} F(v^u)_p \xrightarrow{(p,p)} Fv^u$$

$\leftarrow$ perfect as $\mathcal{O}_v/p$ semiperfect

Case distinction:

- if w^u and v^u are comparable $\Rightarrow$ w^u refines $(v^u)_p$

[this is the crucial case!]

$\Rightarrow$ $(v^u)_p$ not unramified
$\Rightarrow$ w^u not unramified
$\Rightarrow$ $w(p)$ not min pos.

$\Rightarrow$ $\mathcal{O}_{w^u}/p$ semiperfect $\Rightarrow$ $\mathcal{O}_w/p$ semiperfect

$\Rightarrow$ $w^u F / w^u(p)$ divisible $\Rightarrow$ $wL / w(p)$ divisible
$\Rightarrow$ wL regular.

- o/w, W^u and V^u have a proper common coarsening with alg. closed residue field.
  $\leadsto$ use a similar argument. $\boxtimes$(2.)

Open question: What is the $\mathcal{L}_{val}$-elem. class generated by all perfectoid fields of characteristic $(0,p)$?