Orbit separation and stratification by isotropy classes of piezoelectricity tensors

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appeared in Journal of Pure and Applied Mathematic (2025)

DDG40 : Structures Algébriques et Ordonnées, Banyuls 2025

Group actions and their invariant rings or fields

$$\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}$$

$\mathcal{V}$ an affine variety defined over $\mathbb{R}$

G an algebraic group defined over $\mathbb{R}$ $\qquad \mathrm{SO}_3(\mathbb{R})$ or $\mathrm{SO}_3(\mathbb{C})$

Action of G on $\mathcal{V}$

$$\begin{array}{ccc} \mathrm{G} \times \mathcal{V} & \rightarrow & \mathcal{V} \\ (g, v) & \mapsto & g \cdot v \end{array} \qquad \text{s.t.} \left\{ \begin{array}{l} 1 \cdot v = v \\ (gh) \cdot v = g \cdot (h \cdot v). \end{array} \right.$$

The orbit of $v \in \mathcal{V}$ is $\mathrm{G} \cdot v = \{g \cdot v \,|\, g \in \mathrm{G}\}$

Polynomial and rational invariants

f in $\mathbb{K}[\mathcal{V}]$ or $\mathbb{K}(\mathcal{V})$ is *invariant* if $f(g \cdot v) = f(v)$ for all $v \in \mathcal{V}$.

The invariant field $\mathbb{K}(\mathcal{V})^{\mathrm{G}}$ is finitely generated.

The invariant ring $\mathbb{R}[\mathcal{V}]^{\mathrm{G}}$ is finitely generated when G is a compact.

Problem 1: Orbit separation

Determine if $v' \notin \mathrm{G} \cdot v$, for given v, $v' \in \mathcal{V}$.

Sufficient condition: $f(v') \neq f(v)$ for some $f \in \mathbb{K}(\mathcal{V})^{\mathrm{G}}$

Theorem

G compact and $\mathbb{R}[\mathcal{V}]^{\mathrm{G}} = \mathbb{R}[f_1, \ldots, f_\kappa]$.
Then

$$v' \in \mathrm{G} \cdot v \;\Leftrightarrow\; f_1(v') = f_1(v), \ldots, f_\kappa(v') = f_\kappa(v).$$

We say that $f_1, \ldots, f_\kappa$ separate G-orbits in $\mathcal{V}$

[Rosenlicht 56]

$\mathbb{C}(\mathcal{V})^{\mathrm{G}} = \mathbb{C}(f_1, \ldots, f_\mu) \quad \Leftrightarrow \quad f_1, \ldots, f_\mu$ separate G-orbits in $\mathcal{V} \setminus \mathcal{W}$.

Often $\kappa \gg \mu$.

Isotropy

G acts on $\mathcal{V}$.

Isotropy group at $v \in \mathcal{V}$

$\mathrm{G}_v = \{g \in \mathrm{G} \mid g \cdot v = v\}$

Isotropy stratum

$\mathcal{V}_{[H]} = \{v \in \mathcal{V} \mid \mathrm{G}_v \in [\mathrm{H}]\}$

Note: $\mathrm{G}_{g \cdot v} = g \mathrm{G}_v g^{-1} \in [\mathrm{G}_v]$

Problem 2: Isotropy stratification

Assume $f_1, \ldots, f_\kappa$ separate G-orbits in $\mathcal{V}$, find
$P = (P_1, \ldots, P_m) \in \mathbb{R}[y_1, \ldots, y_\kappa]$ s.t. $\overline{\mathcal{V}_{[H]}} = \mathcal{V}\left(P(f_1, \ldots, f_\kappa)\right)$

Theorem

G compact.

- The isotropy classes form a poset: $[\mathrm{H}] \prec [\mathrm{H}'] \Leftrightarrow \exists g \in \mathrm{G}\; g\,\mathrm{H}\,g^{-1} \subset \mathrm{H}'$
- $\overline{\mathcal{V}_{[H]}} = \{v \in \mathcal{V} \mid [\mathrm{H}] \prec [\mathrm{G}_v]\}$

Note: $\overline{\mathcal{V}_{[\mathrm{H}]}} = \mathrm{G} \cdot \mathcal{V}_{\mathrm{H}}$ where $\mathcal{V}_{\mathrm{H}} = \{v \in \mathcal{V} \mid \mathrm{G}_v \subset \mathrm{H}\}$

Orbit separation and stratification by isotropy classes

The piezoelectricity tensor as an example

$\mathrm{G} = \mathrm{SO}_3(\mathbb{R})$

$$\mathcal{V} := \{\mathrm{p} \in \mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3 \,|\, \mathrm{p}_{ijk} = \mathrm{p}_{ikj}\}$$

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where

$$\mathcal{H}_d := \left\{ p \in \mathbb{R}[u,v,w]_d \;\middle|\; \frac{\partial^2 p}{\partial u^2} + \frac{\partial^2 p}{\partial v^2} + \frac{\partial^2 p}{\partial w^2} = 0 \right\}.$$

For $p \in \mathbb{R}[u,v,w]_d$ and $g \in \mathrm{SO}_3$, $\quad g \cdot p = p \circ g^{-1}$

The piezoelectricity tensor as an example

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- $\mathcal{H}_1 = \langle u,\ v,\ w \rangle \cong \mathbb{R}^3$
- $\mathcal{H}_2 = \left\{ \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} a_1 & b_3 & b_1 \\ b_3 & a_2 & b_2 \\ b_1 & b_2 & a_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \;\middle|\; a_1 + a_2 + a_3 = 0 \right\} \cong \mathrm{S}_3^o$

The piezoelectricity tensor as an example

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- $\mathcal{H}_1 = \langle u,\, v,\, w \rangle \cong \mathbb{R}^3$
- $\mathcal{H}_2 \cong \mathrm{S}_3^o$ symmetric matrices with trace zero

- $\mathcal{H}_d = \langle X_d^m \,|\, -d \leq m \leq d \rangle$ with $\left\langle X_d^{-m}, X_d^m \right\rangle$ an irreducible representation of $\mathrm{O}_2(\mathbb{R})$.

The piezoelectricity tensor as an example

$\mathrm{G} = \mathrm{SO}_3(\mathbb{R})$

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- $\mathcal{H}_3 = \left\langle \begin{array}{lll} u(2u^2 - 3v^2 - 3w^2), & u(v^2 - w^2), & \\ v(2v^2 - 3u^2 - 3w^2), & v(w^2 - u^2), & uvw, \\ w(2w^2 - 3v^2 - 3u^2), & w(u^2 - v^2) & \end{array} \right\rangle$

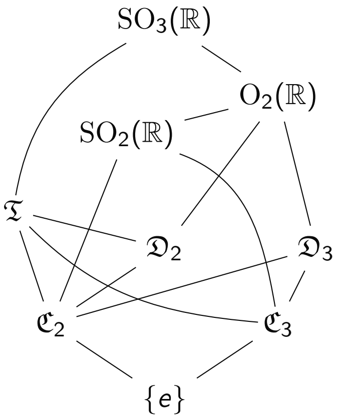
The piezoelectricity tensor as an example

$\mathrm{G} = \mathrm{SO}_3(\mathbb{R})$

$$\mathcal{V} \cong \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$$

[Olive 2014]

$\mathbb{R}[\mathcal{V}]^{\mathrm{G}} = \mathbb{R}[f_1, \ldots, f_{495}]$



Our proposal

$\mathrm{G} = \mathrm{SO}_3(\mathbb{R})$

1. Construct a *separating system*

$$\{(\mathcal{V}_0, \tilde{F}_0), (\mathcal{V}_1, \tilde{F}_1), (\mathcal{V}_2, \tilde{F}_2), (\mathcal{V}_3, \tilde{F}_3), (\mathcal{V}_4, \tilde{F}_4)\}$$

such that

- $\mathcal{V} = \mathcal{V}_0 \sqcup \mathcal{V}_1 \sqcup \mathcal{V}_2 \sqcup \mathcal{V}_3 \sqcup \mathcal{V}_4$
- $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4$ are SO_3-invariant locally closed sets
- $\tilde{F}_i \subset \mathbb{R}[\mathcal{V}_i]^{\mathrm{G}}$ separates obits in $\mathcal{V}_i$

Methodology:

- $\mathcal{V}_i = \mathrm{G} \cdot \mathcal{Z}_i$
- $\mathcal{Z}_i$ is N_i-slice in $\partial\mathcal{V}_{i-1}$
- $F_i \in \mathbb{R}[\mathcal{Z}_i]^{\mathrm{N}_i}$ separates N_i-orbits in $\mathcal{Z}_i$
- $\tilde{F}_i$ is the lifted set to $\mathbb{R}[\mathcal{V}_i]^{\mathrm{G}}$

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2. Isotropy strata within each $\mathcal{V}_i$

Within each $\mathcal{V}_i$ determine the relationships on the $\tilde{F}_i$ that cut the closures of the isotropy strata.

The problem shall be reduced to determine the isotropy strata for the action of a subgroup N_i acting on a subvariety $\mathcal{Z}_i$

Final stratification over the reals

- $\overline{\mathcal{V}_0} = \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 = \mathcal{V}$
  40 invariants separate orbits outside of $\mathcal{V}(x_{-1}^2 + x_0^2 + x_1^2)$
- $\overline{\mathcal{V}_1} = \{0\} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$
  28 invariants separate orbits outside of $\mathcal{V}(x_{-1}^2 + x_0^2 + x_1^2)$

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- $\overline{\mathcal{V}_2} = \{0\} \oplus \{0\} \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ $\quad$ $\mathcal{H}_2 \cong$ symmetric matrices with trace zero
  17 invariants separate orbits outside of $\mathcal{V}\left(\mathrm{discrim}(\chi_A)\right)$,
  i.e., matrices with a double eigenvalue.
- $\overline{\mathcal{V}_3} = \{0\} \oplus \{0\} \oplus \mathcal{B}_2 \oplus \mathcal{H}_3$ $\quad$ $\mathcal{B}_2 = \{A \in \mathcal{H}_2 \mid \deg \mathrm{minpoly}(A) \leq 2\}$
  12 invariants separate orbits outside of $\mathcal{V}(A^2)$, *i.e.*, nilpotent matrices.

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  12 invariants separate orbits outside of $\mathcal{V}(A^2)$, *i.e.*, nilpotent matrices.
- $\overline{\mathcal{V}_4} = \{0\} \oplus \{0\} \oplus \{0\} \oplus \mathcal{H}_3$
  5 invariants generate $\mathbb{R}[\mathcal{H}_3]^{\mathrm{SO}_3(\mathbb{R})}$ [Smith & Bao, 1997]

Total: 102 invariants, and at most 40 need to be evaluated.

Orbit separation and stratification by isotropy classes

Seshadri slice

G an algebraic group acting on $\mathcal{V}$.

A subvariety $\mathcal{S} \subset \mathcal{V}$ is a N-**slice** if

- $N = \{g \in G \mid g \cdot \mathcal{S} \subset \mathcal{S}\}$
- $\mathcal{V} = \overline{\mathrm{G} \cdot \mathcal{S}}$
- $\mathrm{G} \cdot z \cap \mathcal{S} = \mathrm{N} \cdot z$, for z in some open subset $\mathcal{Z}$ of $\mathcal{S}$

$$f \in \mathbb{C}(\mathcal{V})^{\mathrm{G}} \quad \Rightarrow \quad f_{|\mathcal{S}} \in \mathbb{C}(\mathcal{S})^{\mathrm{N}}$$

The slice lemma [Sheshadri 62]

The restriction of rational functions on $\mathcal{V}$ to $\mathcal{S}$ gives the field isomorphism:

$$\mathbb{C}(\mathcal{V})^{\mathrm{G}} \xrightarrow{\cong} \mathbb{C}(\mathcal{S})^{\mathrm{N}}.$$

Example: O_3 on ternary quadrics

Action:

$$\begin{array}{ccc} \mathrm{O}_3(\mathbb{C}) \times \mathrm{S}_3(\mathbb{C}) & \to & \mathrm{S}_3(\mathbb{C}) \\ (Q, A) & \mapsto & Q^t\, A\, Q \end{array}$$

Slice: Diagonal matrices

$$\Lambda = \left\{ \begin{pmatrix} \lambda_1 & \cdot & \cdot \\ \cdot & \lambda_2 & \cdot \\ \cdot & \cdot & \lambda_3 \end{pmatrix} \right\}$$

For a symmetric matrix A s.t. $\mathrm{disc}(\chi_A) \neq 0$ there exists $Q \in \mathrm{G}$ s.t $Q\,A\,Q^T \in \Lambda$.

Additionally: Λ invariant under $\mathfrak{O} = \mathfrak{S}_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$

$Q^t\,\Lambda\,Q \subset \Lambda$ if Q is a permutation matrix or $Q = \begin{pmatrix} \pm1 & \cdot & \cdot \\ \cdot & \pm1 & \cdot \\ \cdot & \cdot & \pm1 \end{pmatrix}$

Thus: The restriction of an $\mathrm{O}_3(\mathbb{C})$-invariant to Λ is a $\mathfrak{O}$-invariant

$\mathrm{Tr}(A)$	$\lambda_1 + \lambda_2 + \lambda_3$
$\mathrm{Tr}(A^2)$	$\lambda_1^2 + \lambda_2^2 + \lambda_3^2$
$\mathrm{Tr}(A^3)$	$\lambda_1^3 + \lambda_2^3 + \lambda_3^3$

The slice lemma [Sheshadri 62]

$$\mathbb{C}(\mathrm{S}_3)^{\mathrm{O}_3} \xrightarrow{\cong} \mathbb{C}(\Lambda)^{\mathfrak{O}}.$$

Normal Seshadri slices & separation

G an algebraic group acting on $\mathcal{V}$.

A subvariety $\mathcal{S} \subset \mathcal{V}$ is a **normal** N-slice if

- $N = \{g \in G \mid g \cdot \mathcal{S} \subset \mathcal{S}\}$
- $\mathcal{V} = \overline{\mathrm{G} \cdot \mathcal{S}}$
- $g \cdot z \in \mathcal{Z} \;\Rightarrow\; g \in \mathrm{N}$, for z in some open subset $\mathcal{Z}$ *normal* in $\mathcal{V}$

For $f \in \mathbb{C}(\mathcal{S})^{\mathrm{N}}$ there is a unique $\tilde{f} \in \mathbb{C}(\mathcal{V})^{\mathrm{G}}$ s.t. $\tilde{f}_{|\mathcal{S}} = f$

[Popov 1992]

f regular on $\mathcal{Z}$ implies $\tilde{f}$ regular on $\mathrm{G} \cdot \mathcal{Z}$

Separation

$F \subset \mathbb{C}[\mathcal{Z}]^{\mathrm{N}}$ separates N-orbits in $\mathcal{Z}$ $\Rightarrow$ $\tilde{F}$ separates G-orbits in $\mathrm{G} \cdot \mathcal{Z}$

Normal Seshadri slices & isotropy stratification

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A subvariety $\mathcal{S} \subset \mathcal{V}$ is a **normal** N-slice if

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- $\mathcal{V} = \overline{\mathrm{G} \cdot \mathcal{S}}$
- $g \cdot z \in \mathcal{Z} \;\Rightarrow\; g \in \mathrm{N}$, for z in some open subset $\mathcal{Z}$ *normal* in $\mathcal{V}$

$\mathrm{G}_z \subset \mathrm{N}_z$ for $z \in \mathcal{Z}$ and thus $[\mathrm{G}_v] \prec [\mathrm{N}]$ for all $v \in \mathrm{G} \cdot \mathcal{Z}$.

Isotropy stratification

Assume $f_1, \ldots, f_\kappa$ separate N-orbits and $\mathcal{Z}$ and for $\mathrm{H} \subset \mathrm{N}$

$$z \in \overline{\mathcal{Z}_{[\mathrm{H}]}} \;\Leftrightarrow\; P_i(f_1, \ldots, f_\kappa) = 0, 1 \leq i \leq n.$$

Then

$$v \in \overline{(\mathrm{G} \cdot \mathcal{Z})_{[\mathrm{H}]}} \;\Leftrightarrow\; P_i(\tilde{f}_1, \ldots, \tilde{f}_\kappa) = 0, 1 \leq i \leq n.$$

Over the reals: field isomorphism and separation

If G, $\mathcal{V}$, N and $\mathcal{Z}$ are all well defined over $\mathbb{R}$

$$\mathbb{R}(\mathcal{V})^{\mathrm{G}_\mathbb{R}} \xrightarrow{\cong} \mathbb{R}(\mathcal{S})^{\mathrm{N}_\mathbb{R}}$$

For $f \in \mathbb{R}(\mathcal{S})^{\mathrm{N}_\mathbb{R}}$ there is a unique $\tilde{f} \in \mathbb{R}(\mathcal{V})^{\mathrm{G}_\mathbb{R}}$ s.t. $\tilde{f}_{|\mathcal{S}_\mathbb{R}} = f$

$F \subset \mathbb{R}[\mathcal{Z}]^{\mathrm{N}_\mathbb{R}}$ separates $\mathrm{N}_\mathbb{R}$-orbits in $\mathcal{Z}_\mathbb{R}$
$\Rightarrow \tilde{F} \subset \mathbb{R}(\mathcal{V})^{\mathrm{G}}$ separates $\mathrm{G}_\mathbb{R}$-orbits in $\mathrm{G}_\mathbb{R} \cdot \mathcal{Z}_\mathbb{R}$

We will be looking at:

- $\mathcal{V}$ and $\mathcal{Z}$ linear spaces, except for $\mathcal{B}_2 \oplus \mathcal{H}_3$

$$\mathcal{B}_2 = \{A \in \mathcal{H}_2 \mid \deg \mathrm{minpoly}(A) \leq 2\}$$

- $\mathrm{G} = \mathrm{SO}_3(\mathbb{C})$, $\mathrm{G}_\mathbb{R} = \mathrm{SO}_3(\mathbb{R})$
- $\mathrm{N} = \mathfrak{O}$ or $\mathrm{N} = \mathrm{O}_2(\mathbb{C})$, $\mathrm{N}_\mathbb{R} = \mathrm{O}_2(\mathbb{R})$

Orbit separation and stratification by isotropy classes

Normal slice for SO_3 acting on $\mathcal{H}_2 \oplus \mathcal{H}$

$$\mathcal{H}_2 = \left\{ [u\ v\ w] \begin{bmatrix} \lambda_1 & \mu_3 & \mu_1 \\ \mu_3 & \lambda_2 & \mu_2 \\ \mu_1 & \mu_2 & \lambda_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \;\middle|\; \lambda_1 + \lambda_2 + \lambda_3 = 0 \right\} \cong \mathrm{S}_3^o$$

$$\mathcal{Z}_2 = \left\{ [u\ v\ w] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \;\middle|\; \begin{array}{l} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \neq 0 \end{array} \right\}$$

$$\mathfrak{O}^+ = \left\langle \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle \cap \mathrm{SO}_3$$

Theorem

$\mathcal{Z}_2 \oplus \mathcal{H}$ is a normal $\mathfrak{O}^+$-slice for $\mathcal{H}_2 \oplus \mathcal{H}$ under the action of SO_3.

Furthermore $\mathcal{H}_2 \oplus \mathcal{H} \setminus \mathrm{G} \cdot (\mathcal{Z}_2 \oplus \mathcal{H}) = \mathcal{V}(\mathrm{discrim}(\chi_A))$.

Separation for SO_3 acting on $\mathcal{H}_2 \oplus \mathcal{H}_3$

$$\mathcal{Z}_2 = \left\{ \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2 \;\middle|\; \begin{array}{l} \lambda_1+\lambda_2+\lambda_3 = 0 \\ (\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1) \neq 0 \end{array} \right\} \subset \mathcal{H}_2$$

$$\mathcal{H}_3 = \left\{ \begin{array}{l} \alpha_1 u(2u^2-3v^2-3w^2) + \alpha_2 v(2v^2-3u^2-3w^2) + \alpha_3 w(2w^2-3v^2-3u^2) \\ +\beta_1 u(v^2-w^2) + \beta_2 v(w^2-u^2) + \beta_3 w(u^2-v^2) + \gamma uvw \end{array} \right\}$$

[H. & Jalard 2025]

Let $[\lambda] = (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)$.

The $\mathfrak{O}^+$-orbits in $\mathcal{Z}_2 \oplus \mathcal{H}_3$ are separated by the entries of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \alpha_1^2 & \beta_1^2 & [\lambda]\alpha_1\beta_1 \\ \lambda_2 & \alpha_2^2 & \beta_2^2 & [\lambda]\alpha_2\beta_2 \\ \lambda_3 & \alpha_3^2 & \beta_3^2 & [\lambda]\alpha_3\beta_3 \end{pmatrix}$$

and $\qquad [\lambda]\gamma,$

$$[\lambda]\alpha_1\alpha_2\alpha_3,\; \beta_1\alpha_2\alpha_3+\beta_2\alpha_1\alpha_3+\beta_3\alpha_1\alpha_2,\; [\lambda]\left(\alpha_1\beta_2\beta_3+\alpha_2\beta_1\beta_3+\alpha_3\beta_1\beta_2\right),\; \beta_1\beta_2\beta_3.$$

Separation for SO_3 acting on $\mathcal{H}_2 \oplus \mathcal{H}_3$

$$\mathbb{R}(\mathcal{Z}_2 \oplus \mathcal{H}_3)^{\mathfrak{O}^+} \cong \mathbb{R}(\mathcal{H}_2 \oplus \mathcal{H}_3)^{\mathrm{SO}_3}$$

Corollary

The *lifts* of the above $\mathfrak{O}^+$-invariants on $\mathcal{Z}_2 \oplus \mathcal{H}_3$ to SO_3-invariants on $\mathcal{H}_2 \oplus \mathcal{H}_3$ separate SO_3-orbits of $\mathcal{H}_2 \oplus \mathcal{H}_3$ outside $\mathcal{V}\left(\mathrm{discrim}(\chi_A)\right)$.

$$\mathrm{discrim}(\chi_A)_{|\mathcal{Z}} = (\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2$$

To evaluate numerically the lifted invariants at $h_2 + h_3 \in \mathcal{H}_2 \oplus \mathcal{H}_3$:

- compute a $g \in \mathrm{SO}_3$ that diagonalizes the matrices associated to h_2
- evaluate the $\mathfrak{O}^+$-invariants at $g \cdot h_2 + g \cdot h_3 \in \mathcal{Z}_2 + \mathcal{H}_3$

Orbit separation and stratification by isotropy classes

Normal slice for SO_3 acting on $\mathcal{B}_2 \oplus \mathcal{H}$

$$\mathcal{B}_2 = \{A \in \mathrm{S}_3^o \mid I,\ A,\ A^2 \text{ linearly dependent } \}$$

$$\mathcal{Z}_3 = \left\{ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\,\lambda \end{bmatrix} \middle| \lambda \in \mathbb{C} \setminus \{0\} \right\}$$

$$\mathrm{N}_3 = \left\{ \left[\begin{array}{c|c} g & 0 \\ \hline 0 & \det g \end{array} \right] \middle| \ g \in \mathrm{O}_2 \right\}$$

Theorem

$\mathcal{Z}_3 \oplus \mathcal{H}$ is a normal O_2-slice for $\mathcal{B}_2 \oplus \mathcal{H}$ under the action of SO_3.

Furthermore $\mathcal{B}_2 \oplus \mathcal{H} \setminus \mathrm{G} \cdot (\mathcal{Z}_3 \oplus \mathcal{H}) = \mathcal{V}(A^2)$ (nilpotent matrices)

Separation for SO_3 acting on $\mathcal{B}_2 \oplus \mathcal{H}_3$

$$\mathcal{H}_3(\mathbb{R}) = \{y_{-3,3} Y_3^{-3} + \ldots + y_{3,3} Y_3^3 \mid y_{-m,3} = \bar{y}_{m,3}\}$$

[Jalard 2025]

The following O_2-invariants separate O_2-orbits in $\mathcal{Z}_3 \oplus \mathcal{H}_3$:

$$\begin{array}{ll}
(y_{2,3} y_{-1,3}^2 - y_{-2,3} y_{1,3}^2)(y_{3,3} y_{-1,3}^3 - y_{-3,3} y_{1,3}^3) & \lambda, \quad y_{0,3}^2, \\
y_{-m,3}^{\frac{m \wedge m'}{m}} y_{m',3}^{\frac{m \wedge m'}{m'}} + y_{m,3}^{\frac{m \wedge m'}{m}} y_{-m',3}^{\frac{m \wedge m'}{m'}} & 1 \leq m \leq m' \leq 3 \\
\mathrm{i}\, y_{0,3} \left(y_{-m,3}^{\frac{m \wedge m'}{m}} y_{m',3}^{\frac{m \wedge m'}{m'}} - y_{m,3}^{\frac{m \wedge m'}{m}} y_{-m',3}^{\frac{m \wedge m'}{m'}} \right), & 1 \leq m < m' \leq 3
\end{array}$$

$$\mathbb{R}\left(\mathcal{B}_2 \oplus \mathcal{H}_3\right)^{\mathrm{SO}_3} \cong \mathbb{R}\left(\mathcal{Z}_3 \oplus \mathcal{H}_3\right)^{\mathrm{O}_2}$$

Corollary

The *lifts* of the above separate SO_3-orbits in $(\mathcal{B}_2 \setminus \mathcal{N}_2) \oplus \mathcal{H}_3$

$\mathcal{N}_2$: symmetric nilpotent matrices

Orbit separation and stratification by isotropy classes

Normal slice for SO_3 acting on $\mathcal{H}_1 \oplus \mathcal{H}$

$$\mathcal{H}_1 = \{x\,u + y\,v + z\,w \mid x, y, z \in \mathbb{R}\}$$

$$\mathcal{Z}_1 = \{\lambda\,w \mid \lambda \in \mathbb{R} \setminus \{0\}\}\}$$

Theorem

$\mathcal{Z}_1 \oplus \mathcal{H}$ is a normal O_2-slice for $\mathcal{H}_1 \oplus \mathcal{H}$ under the action of SO_3.

Furthermore $\mathcal{H}_1 \oplus \mathcal{H} \setminus \mathrm{G} \cdot (\mathcal{Z}_1 \oplus \mathcal{H}) = \mathcal{V}(x^2 + y^2 + z^2)$

Separation for SO_3 acting on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$

$$\mathcal{H}_d = \left\{ y_{-d,d} Y_d^{-d} + \ldots + y_{d,d} Y_d^d \,|\, y_{-m,d} = \bar{y}_{m,d} \right\}$$

[Jalard 2025]

The following O_2-invariants separate O_2-orbits on $\mathcal{Z}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$

$$y_{0,2}, \quad y_{0,1}\, y_{0,d}, \qquad d \in \{1,3\}$$

$$y_{-m,d}^{\frac{m\wedge m'}{m}} y_{m',d'}^{\frac{m\wedge m'}{m'}} + y_{m,d}^{\frac{m\wedge m'}{m}}\, y_{-m',d'}^{\frac{m\wedge m'}{m'}} \qquad \begin{cases} d \in \{2,3\}, \\ m \geq 1, \end{cases} \quad (m,d) \leq (m',d')$$

$$\mathrm{i}\, y_{0,1} \left(y_{-m,d}^{\frac{m\wedge m'}{m}} y_{m',d'}^{\frac{m\wedge m'}{m'}} - y_{m,d}^{\frac{m\wedge m'}{m}}\, y_{-m',d'}^{\frac{m\wedge m'}{m'}} \right), \qquad \begin{cases} d \in \{2,3\}, \\ m \geq 1, \end{cases} \quad (m,d) \leq (m',d')$$

$$\mathbb{R}\left(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3\right)^{\mathrm{SO}_3} \cong \mathbb{R}\left(\mathcal{Z}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3\right)^{\mathrm{O}_2}$$

Corollary

The *lifts* of the above separate SO_3-orbits in $(\mathcal{H}_1 \setminus \mathcal{V}(x^2+y^2+z^2)) \oplus \mathcal{H}_3$

Generalisations

We can explicitely deal with the SO_3 representations

$$\mathcal{V} = (\mathcal{H}_1)^m \oplus (\mathcal{H}_2)^n \oplus \mathcal{H}_3 \quad \text{or} \quad (\mathcal{H}_1)^m \oplus (\mathcal{H}_2)^n \oplus \mathcal{H}_4$$

as for instance

- the elasticity tensor $\mathcal{H}_2 \oplus \mathcal{H}_2 \oplus \mathcal{H}_4$
- $\mathbb{R}[u, v, w]_3 \cong \mathcal{H}_1 \oplus \mathcal{H}_3$
- $\mathbb{R}[u, v, w]_4 \cong \mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \mathcal{H}_4$

THANKS!

Orbit separation and stratification by isotropy classes of piezoelectricity tensors. H. & Jalard, Journal of Pure and Applied Algebra (2025)

More with Seshadri slices:

- Invariants of the action of SO_3 or O_3 on $\mathcal{H}_1 \oplus \mathcal{V}$ and 3D shape recognition
  H. & Jalard (2025)
- Invariants of the action of O_3 on $\mathbb{K}[x, y, z]_{2d}$ and neuroimaging
  Görlach, H. & Papadopoulo, Foundations of Computational Math. (2019)
- Rationality of the invariant field of SO_n on $\mathcal{H}_1^{\nu} \oplus \mathcal{V}$
  H. & Jalard, Journal of Algebra (2025)