Intro : Definable Types and Stable Embeddedness

Fix T complete $\mathcal{L}$-theory with QE, $M \preceq \mathcal{U} \models T$, $\mathcal{U}$ suff saturated.

Recall: * $p(x) \in S_x(M)$ is **definable** if for any $\mathcal{L}$-fml $\varphi(x,y)$ there is an $\mathcal{L}_M$-fml $d_p\varphi(y)$ s.th for any $b \in M^y$ one has $\varphi(x,b) \in p$ iff $\models d_p\varphi(b)$.

* Let $M \subseteq A \subseteq \mathcal{U}$. We say M is **stably embedded in A** if for any $\mathcal{L}_A$-fml $\psi(x)$ the set $\psi(\mathcal{U}) \cap M^x = \{a \in M^x \mid \mathcal{U} \models \psi(a)\}$ is definable in M.
We will write $M \subseteq_{SE} A$ and say (A, M) is an **SE-pair**.

RK: For $M \subseteq A \subseteq \mathcal{U}$, M is stably embedded in A
$\Leftrightarrow$ for any finite tuple a of elements from A, $tp(a/M)$ is definable.

Fact (Shelah) T is stable iff all types are definable, over all $M \models T$.

Some Prehistory: Early Results on Definable Types in Unstable Theories.

Absolute case

* (van den Dries) If $\mathcal{R} = \langle \mathbb{R}, <, \ldots \rangle$ is **o-minimal** (e.g. the real field or $\mathcal{R} = \mathbb{R}_{an,exp}$), then all types over $\mathcal{R}$ are definable ($\Leftrightarrow \mathcal{R} \subseteq_{SE} \mathcal{U}$)

** (folklore) $T = \underline{PRES}$: $\mathbb{Z}$ is the only model (up to isomorphism) over which all types are definable.

*** (Delon 1989) $T = \underline{pCF}$: $\mathbb{Q}_p$ is the only model over which all types are definable.

Relative case

* (Marker–Steinhorn 1994) T dense o-minimal, $M \preceq N \models T$, then $M \subseteq_{SE} N$ iff M is Dedekind complete in N.

** (folklore) $M \preceq N \models PRES$, then $M \subseteq_{SE} N$ iff N is an end extension of M.

Algebraic Characterization of Def Types in Benign Valued Fields

<u>Recall</u>: An extension L/K of valued fields is **separated** if every finite dim K-sub-vector space admits a basis $(b_1, \ldots, b_n)$ s.th. for all $\bar{\alpha} \in K^n$: $\mathrm{val}(\sum_{i=1}^n \alpha_i b_i) = \min_{i=1}^n \mathrm{val}(\alpha_i b_i)$

<u>Fact</u>: Let $K \subseteq L \subseteq \tilde{K}$ be valued fields with $K \preceq \tilde{K}$. Then

1. (Cubides-Delon 2016) If $K \models ACVF$, then $K \subseteq_{SE} L$ iff $[L/K$ is separated and $\Gamma_K \subseteq_{SE} \Gamma_L]$
2. (Cubides-Ye 2021) If $K \models RCVF$ or $K \models pCF$, then $K \subseteq_{SE} L$ iff $[L/K$ is separated, $k_K \subseteq_{SE} k_L$ and $\Gamma_K \subseteq_{SE} \Gamma_L]$
3. (Touchard 2024) If K is henselian of char $(0,0)$, or alg maximal Kaplansky of char (p,p) or finitely ramified henselian with perfect residue field, then $K \subseteq_{SE} L$ iff $[L/K$ is separated and $RV_K \subseteq_{SE} RV_L]$

In case $L \not\subset K$, this is further equivalent to $[L/K$ separated, $k_K \subseteq_{SE} k_L$ and $\Gamma_K \subseteq_{SE} \Gamma_L]$

The source for the reduction : Domination

The following powerful principle was (essentially) observed by Haskell-Hrushovski-Macpherson

Fact ("algebraic domination")

Let $K \subseteq L, M \subseteq \tilde{K}$ be valued fields. Assume:

1. L/K is separated.
2. $\Gamma_L \cap \Gamma_M = \Gamma_K$ and $k_L \underset{k_K}{\overset{ld}{\downarrow}} k_M$ ($\overset{ld}{\downarrow}$ means "linearly disjoint")

Then the isomorphism type of the valued field LM is completely determined by the isomorphism types of $\Gamma_L + \Gamma_M$ and $k_L k_M$. Moreover, LM/M is separated, $L \underset{K}{\overset{ld}{\downarrow}} M$, $k_{LM} = k_L k_M$ and $\Gamma_{LM} = \Gamma_L + \Gamma_M$.

Uniform Definability of Definable Types

<u>Def</u>: T has <u>UDDT</u> (uniform definability of definable types) if for any $\mathcal{L}$-fml $\varphi(x,y)$ there is an $\mathcal{L}$-fml $\chi(y,z)$ s.th for every def. type $p(x) \in S_x(M)$ there is $c \in M^z$ s.th $d_p\varphi(y) \equiv \chi(y,c)$

<u>RK</u>: If T has UDDT and X is a definable set, the class of definable types concentrated on X, S^X_{def}, is naturally given by a pro-definable set, i.e. a projective limit of definable sets with def. transition functions.

Indeed, $p(x) \in S^X_{def}(M)$ is coded by $(\ulcorner d_p\varphi \urcorner)_\varphi$, an infinite tuple in M^{eq}, and by UDDT this coding is uniform, so S^X_{def} is a type-def subset of an infinite product of def sets.

<u>Variant</u>: For $\mathcal{C}$ a subclass of all definable types, we say elements of $\mathcal{C}$ are uniformly definable if there are $\chi(y,z)$ as above for all $\varphi(x,y)$ providing definitions for all types in $\mathcal{C}$.

UDDT and the language of pairs

* We will consider SE-pairs in the language $\mathcal{L}_P := \mathcal{L} \cup \{P\}$

$$(A, M) \rightsquigarrow \mathcal{A} := (A, P(A) = M)$$

* Given an $\mathcal{L}_P$-embedding $\mathcal{A} \subseteq \mathcal{B}$ of SE-pairs, we say it is a **bp-embedding**, $\mathcal{A} \subseteq_{bp} \mathcal{B}$, if $tp_{\mathcal{L}}(A/P(B)) = tp_{\mathcal{L}}(A/P(A)) \mid P(B)$

* Def: T has EP (extension property) if every SE-pair bp-embeds into an elementary SE-pair.

Easy observation: T has UDDT iff the class of SE-pairs is axiomatizable (by T_{SE}) in $\mathcal{L}_P$.

Assuming T has EP, T has UDDT iff the class of elementary SE-pairs is axiomatizable (by T_{SE}^{el}) in $\mathcal{L}_P$.

Examples: 1. ACVF, RCVF, pCF have EP and UDDT (Cubides-Ye)
2. A theory of benign valued fields has EP and UDDT provided residue field theory and value group theory do (Touchard)

Strict pro-definability of spaces of def. types

<u>Def</u>: A pro-def. set $D = \varprojlim_{i \in I} D_i$ is called <u>strict prodefinable</u>

if $\pi_i(D) \subseteq D_i$ is definable for every $i \in I$.

<u>Def</u>: Let X be a definable set in a valued field K.

- $\widehat{X}(K) := \{ p \in S^X_{def}(K) \mid p \perp \Gamma \}$
- $\widetilde{X}(K) := \{ p \in S^X_{def}(K) \mid p \perp +\infty \text{ (in } \Gamma) \}$

The corresponding class of definable types are denoted $\mathcal{C}_{\perp\Gamma}$ and $\mathcal{C}_{bdA}$

Motivation for all this:

<u>Fact</u> (Hrushovski-Loeser) In ACVF, $\widehat{X}$ is strict pro-definable.
Moreover, if C is an algebraic curve, $\widehat{C}$ is even definable.

<u>Note</u>: For an alg variety V, $\widehat{V}$ is a model-theoretic analog of the Berkovich analytification V^{an}.

More spaces of definable types are strict pro-definable

Thm (Cubides-H-Ye)

Let X be a definable set in a model of ACVF. Then

① S^X_{def}, $\tilde{X}$ and $\widehat{X}$ are all strict pro-def.

② If C is a curve, S^C_{def}, $\tilde{C}$ and $\widehat{C}$ are all def.

* The proof of ① uses an a priori stronger concept ("beauty transfer") which allows for an AKE Principle.

* For ②, if $C \subseteq \mathbb{A}^n$, using Riemann-Roch and some elementary tricks, we find $d = d(C) \in \mathbb{N}$ s.th any $p(x) \in S^C_{def}(K)$ is determined by fmls of the form $val(f(x)) \leq val(g(x))$, where $f, g \in K[X_1, \dots, X_n]_{\leq d}$.

Beautiful Pairs – Generalities

Def: T has AP (amalgamation property) if the class $(\mathcal{C}_{dy}, \leq_{bp})$ has AP, where $\mathcal{C}_{dy}$ is the class of all SE-pairs.

Fact: There are dp-minimal theories where EP fails (Cubides-H-Ye) and even dp-minimal theories where AP fails (H-Rennani).

Def: An SE-pair $\mathcal{M} = (M, P(\mathcal{M}))$ is a beautiful pair if whenever $A \leq_{bp} \mathcal{M}$ and $A \leq_{bp} B$ with $|B| \leq |T|$, B bp-embeds into $\mathcal{M}$ over A.

Fact (CHY) 1. beautiful pairs exist iff T has AP.

Now assume T has AP. Then

2. Any two b.p. are back-and-forth equivalent. Thus their common theory T_{bp} is complete.

3. $T_{bp} \models T$ iff T has EP.

Beauty transfer and strict pro-definability

Assume T has AP

<u>Def</u>: T has **beauty transfer** if there is a $|T|^+$-saturated b.p.

($\Leftarrow$ all b.p. are $|T|^+$-saturated).

<u>Thm</u> (CHY)

If T has AP and beauty transfer, then T has UDDT and S^X_{def} is strict pro-definable.

<u>RK</u>: This all relativizes to natural subclasses of definable types, e.g. $\mathcal{C}_{bdd}$ or $\mathcal{C}_{\perp\Gamma}$ in valued fields.

Beauty transfer in benign valued fields

<u>Thm</u> (CHY)

① ACVF has AP, EP and beauty transfer, for $\mathcal{C}_{\mathrm{def}}$, $\mathcal{C}_{\mathrm{bdd}}$ and $\mathcal{C}_{\perp\Gamma}$.

② There is an AKE Principle for these properties:
If the theories of the residue field and value group have AP, EP and beauty transfer in a benign valued field, so does the valued field.

②' The content of ② also holds for natural subclasses of def types.

Separably closed valued fields

$SCVF_{p,e}$: $\mathcal{L}_{div}$-theory of sep closed non-trivially valued fields of char $p>0$ and degree of imperfection $e \in \mathbb{N} \cup \{\infty\}$ (i.e $[K:K^p] = p^e$)

Thm (He-H 2025+)
For the classes $\mathcal{C}_{def}$, $\mathcal{C}_{I\Gamma}$, $\mathcal{C}_{bdd}$ of def types in $SCVF_{p,e}$ one has AP, EP and beauty transfer. In each case a concrete axiomatization of the theory of beautiful pairs may be given.

Cor (He-H. 2025+)
Let X be a def set in $K \models SCVF_{p,e}$ (even in the geometric sorts)
Then $S_X(K)$, $\widehat{X}$ and $\widetilde{X}$ are strict pro-definable.

RK: For e finite and $\widehat{X}$ this was shown in 2018. (H-Kamensky-Rideau)

Valued fields with contracting automorphism

For p prime consider the valued difference field $K_p = (\mathbb{F}_p(t)^{alg}, v_t, \mathrm{Frob}_p)$

$$VFA_0 := \{\varphi \ \mathcal{L}_{div} \cup \{\sigma\}\text{-sentence} \mid K_p \models \varphi \text{ for } p \gg 0\}$$

<u>Fact (Hrushovski)</u>
VFA_0 is the model companion of all contractive valued difference fields of equichar 0. (contractive $\equiv$ if $val(x) > 0$, then $val(\sigma(x)) \geq n\, val(x)$ for all $n \in \mathbb{N}$)

$\mathcal{C}^{tr}_{def}$:= class of all def types in VFA_0 which are $\perp$ to all σ-algebraic types in the residue field.

$\mathcal{C}^{tr}_{\perp\Gamma}$ and $\mathcal{C}^{tr}_{bdd}$ are defined accordingly.

<u>Thm</u> (H-Hrushovski-Ye-Zou, WIP) In all 3 cases (above) AP, EP and beauty transfer holds. Thus X^{tr}_{def}, $\hat{X}^{tr}$ and $\tilde{X}^{tr}$ are strict pro-definable for any def set X in VFA_0.