Elimination of quantifiers in the theory of projectable real closed rings with the first convexity property.

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First notions

An f-ring is a subdirect products of totally ordered rings (a lattice ordered ring). This is expressible by an universal first order sentence in the language of lattice-ordered rings $\mathcal{L}_{\mathrm{lor}} = \{0, 1, +, \cdot, \wedge\}$. Precisely:

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An f-ring A is **projectable** if $A = a^{\perp} + a^{\perp\perp}$, for all $a \in A$; where the **polar** of $a \in A$ is $a^{\perp} = \{b \in A : a \perp b\}$ and the **bipolar** of a is $a^{\perp\perp} = \{b \in A : b \perp c \text{ for all } c \in a^{\perp}\}$. This notion is expressed by a first order formula in $\mathcal{L}_{\rm lor}$.

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An ordered ring A satisfies the **first convexity property** if

$$\forall a \forall b (0 < a < b \rightarrow b \mid a),$$

is true in A.

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A lattice ordered ring is called **divisible-projectable** if it satisfies the following formula:

$$\forall x \forall y \Big(y \neq 0 \to \exists z \exists w \big(x = z + w \wedge z \perp w \wedge y \mid z \wedge$$

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A ring A is called **sc-regular** if there exists an element $u \in A$ such that $\mathrm{Ann}(u) = \{0\}$ (or $1 \preceq u$) and $u \nmid e$ for every non-zero idempotent $e \in A$.

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In the context of this talk, you may think real closedness equivalent to the satisfaction of the intermediate value property for polynomials.

Cherlin-Dickmann theorem

Cherlin and Dickmann proved (1983) that the theory of **real closed valuation rings** admits elimination of quantifiers in the language $\mathcal{L}_{\mathrm{or}} \cup \{|\}$, where $|$ is the divisibility relation.

Boolean Products

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We say that $\mathfrak{A}$ is a **Boolean product** of $\{\mathfrak{A}_x : x \in X\}$ in L, denoted by $\mathfrak{A} \in \Gamma_L^a\big(X, (\mathfrak{A}_x)_{x \in X}\big)$, if the following conditions holds:

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(i) X is a Boolean space.

(ii) $\mathfrak{A}$ is a subdirect product of $\{\mathfrak{A}_x : x \in X\}$.

(iii) For every atomic L-formula $\Phi(v_1, \dots, v_n)$ and every $a_1, \dots, a_n \in |\mathfrak{A}|$,

$$\big[\!\!\big[\, \Phi(a_1, \dots, a_n) \,\big]\!\!\big] =_{\mathrm{def}} \big\{x \in X : \mathfrak{A}_x \models \Phi\big(a_1(x), \dots, a_n(x)\big)\big\}$$

is a clopen subset of X.

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(iv) Patchwork property: For every $a, b \in \mathfrak{A}$ and any clopen set N of X, the element $c = a_{\restriction_N} \cup b_{\restriction_{X \setminus N}}$ defined by

$$c(y) = \begin{cases} a(y) & \text{if } y \in N \\ b(y) & \text{if } y \in X \setminus N, \end{cases}$$

belongs to $|\mathfrak{A}|$.

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We say that $\mathfrak{A}$ is an **elementary Boolean product** of $\{\mathfrak{A}_x : x \in X\}$ in L, denoted by $\mathfrak{A} \in \Gamma^e_L\big(X, (\mathfrak{A}_x)_{x\in X}\big)$, if $\mathfrak{A}$ is a Boolean product of $\{\mathfrak{A}_x : x \in X\}$ in L and condition (iii) is verified for **all** L-formulas $\Phi(v_1, \ldots, v_n)$.

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Let A be a reduced f-ring. Then A is a projectable, divisible-projectable, sc-regular real closed ring with the first convexity property if and only if A is the ring of continuous sections of a Haussdorff sheaf of Real Closed Valuation Rings (in the language of ordered rings adjoining the divisibility relation), that is

$$A \in \Gamma^e_{\mathcal{L}_{\mathrm{or}}\cup\{|\}}\big(\pi A, (A/p)_{p\in\pi A}\big),$$

where πA is the space of minimal prime ideals of A and A/p is a real closed valuation ring, for every $p \in \pi A$.

Radical relations

Radical relations were defined by Prestel-Schwartz in the following way:

(1) $a \preceq a$, for all $a \in A$;

(2) if $a \preceq b$ and $b \preceq c$ then $a \preceq c$, for all $a, b, c \in A$;

(3) if $a \preceq c$ and $b \preceq c$ then $a + b \preceq c$, for all $a, b, c \in A$;

(4) if $a \preceq b$ then $ac \preceq bc$, for all $a, b, c \in A$;

(5) $a \preceq 1$, for all $a \in A$ and $1 \not\preceq 0$;

(6) $b \preceq b^2$, for all $b \in A$.

Radical relations and von Neumann regular rings

Prestel-Schmid (1990) showed that for any radical relation $\preceq$ on a ring A, there exists a subset $X \subseteq \mathrm{Spec}(A)$ such that:

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• if $X = \pi A$ is the space of minimial prime ideals of a ring A, then the radical relation given by X is

$$a \preceq_{\pi A} b \iff \forall x (bx = 0 \to ax = 0) \iff \mathrm{Ann}(b) \subseteq \mathrm{Ann}(a).$$

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• the theory of von Neumann regular real closed rings without non-zero minimal idempotents admits elimination of quantifiers in the $\mathcal{L}_{\mathrm{or}} = \{0, 1, +, \cdot, \wedge, \preceq\}$.

A theory of real closed rings and Local Divisibility

Let T^* be the theory of projectable, sc-regular, divisible-projectable, real closed rings without minimal (non-zero) idempotents.

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The local divisibility relation is defined by:

$$y \mid_{\rm loc} w \longleftrightarrow \exists w'(w' \neq 0 \wedge w'(w - w') = 0 \wedge y \mid w') \vee w = 0,$$

or equivalentely:

$$y \mid_{\rm loc} w \longleftrightarrow \exists e\big[e^2 = e \wedge we \neq 0 \wedge e \preceq w \wedge y \mid we\big] \vee w = 0.$$

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Theorem (G., 2025): The theory T^* admits elimination of quantifiers in language $\mathcal{L} = \{0, 1, +, \cdot, \wedge, |, \preceq, |_{\rm loc}\}$.

Idea of the proof

The model completeness of the theory T^* is established in the language of $\{0, 1, +, \cdot, \wedge, \preceq, |_{\mathrm{loc}}\}$ using Comer's result (a Feferman-Vaught result for Boolean products) and the fact that global morphisms respecting local divisibility entails local morphisms respecting the usual divisibility (a similar result for q. e. will be soon stated).

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In order to prove the elimination of quantifiers of T^*, I will first describe the universal theory $T^*_\forall$ and secondly, prove that the amalgamation property for this theory $T^*_\forall$.

Universal theory

Theorem (G., 2025): The universal theory of T^* in the language $\{0, 1, +, \cdot, \wedge, |, \preceq, |_{\mathrm{loc}}\}$ is the theory of reduced f-rings satisfying the following axioms:

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• the glueing axioms for divisibility:
$\forall a \forall b \forall c_1 \cdots \forall c_n \big((bc_1 - a) \cdots (bc_n - a) = 0 \rightarrow b \mid a\big)$, for each $n \in \mathbb{N}$.

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• the local divisibility property: $\forall a \forall b \forall c (a \not\preceq bc - a \rightarrow b \mid_{\mathrm{loc}} a)$.

Amalgamation property

Let $A, B, C \models T_\forall^*$ such that there exists $f\colon A \to B$ and $g\colon A \to C$ monomorphisms in the language $\{0, 1, +, \cdot, \wedge, |, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$.

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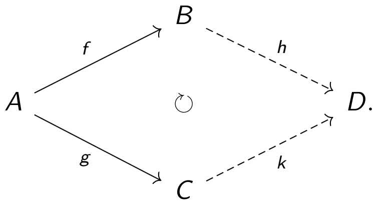
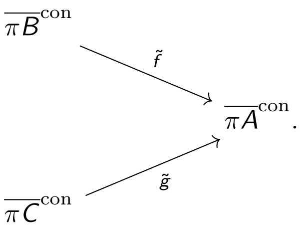
We want to find $D \models T^*_\forall$ and monomorphisms $h\colon B \to D$ and $k\colon C \to D$ such that the following diagram is commutative:



Amalgamation property

By **Prestel-Schwartz**, there exists $\tilde{f}\colon \overline{\pi B}^{\mathrm{con}} \to \overline{\pi A}^{\mathrm{con}}$ and $\tilde{g}\colon \overline{\pi C}^{\mathrm{con}} \to \overline{\pi A}^{\mathrm{con}}$ continuous and surjective functions; that is:



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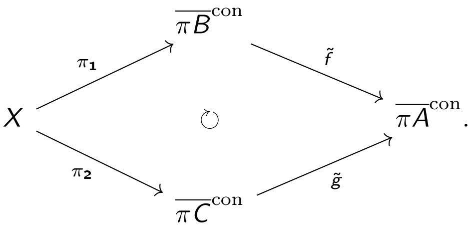
$$\begin{array}{ccc} \overline{\pi B}^{\mathrm{con}} & & \\ & \overset{\tilde{f}}{\searrow} & \\ & & \overline{\pi A}^{\mathrm{con}}. \\ & \underset{\tilde{g}}{\nearrow} & \\ \overline{\pi C}^{\mathrm{con}} & & \end{array}$$

This diagram can be completed using the coproduct of $\overline{\pi B}^{\mathrm{con}}$ and $\overline{\pi C}^{\mathrm{con}}$ over $\overline{\pi A}^{\mathrm{con}}$, given by:

$$X = \overline{\pi B}^{\mathrm{con}} \times_{\overline{\pi A}^{\mathrm{con}}} \overline{\pi C}^{\mathrm{con}} = \left\{(q_1, q_2) \in \overline{\pi B}^{\mathrm{con}} \times \overline{\pi C}^{\mathrm{con}} : \tilde{f}(q_1) = \tilde{g}(q_2)\right\}.$$

Amalgamation property and local morphisms

Thus we have the following commutative diagram:



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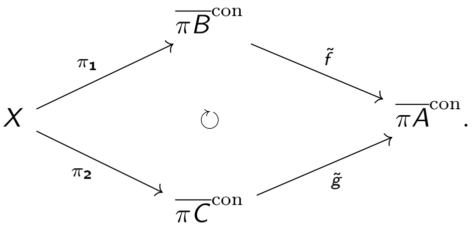

Theorem (G., 2025): Let $A \models T^*_\forall$ in the language $\{0, 1, +, \cdot, \wedge, |, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$ and let $B \models T^*$. Let $f : A \to B$ a monomorphism in the language $\{0, 1, +, \cdot, \wedge, |, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$. For $q \in \overline{\pi B}^{\mathrm{con}}$ and $p \in \overline{\pi A}^{\mathrm{con}}$ such $\tilde{f}(q) = f^{-1}(q) = p$, one has that $f_{pq} : A/p \to B/q$ is a monomorphism in the language $\mathcal{L}' = \{0, 1, +, \cdot, \leqslant, |\}$.

Maximal local divisibility

In order to prove the previous theorem, the notion of local divisibility had to be strengthened to what I called a maximal local divisibility:

$$b \mid^{\mathrm{m}}_{\mathrm{loc}} a \leftrightarrow a = 0 \vee \exists e \Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae$$

$$\wedge \ \big(a(1-e) = 0 \rightarrow b \mid a\big) \wedge \Big(a(1-e) \neq 0 \rightarrow$$

$$\forall f \left(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \rightarrow b \nmid a(1-e)f\right)\Big)\Big].$$

Using the previous theorem for $\mathfrak{q} = (q_1, q_2) \in X$ and putting $p = \tilde{f}(q_1) = \tilde{g}(q_2) \in \overline{\pi A}^{\mathrm{con}}$, one has monomorphisms $f_{pq_1} : A/p \to B/q_1$ and $g_{pq_2} : A/p \to C/q_2$ in the language $\mathcal{L}' = \{0, 1, +, \cdot, \leqslant, |\}$.

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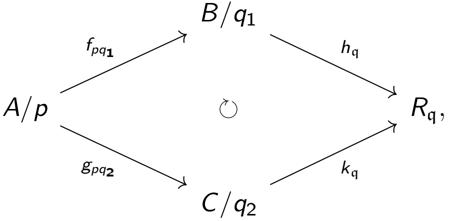
By quantifier elimination of the theory RCVR, there exists a real closed valuation ring $R_\mathfrak{q}$ such that the following diagram is conmutative:



Amalgamation property

Now consider the ring $D = \prod_{\mathfrak{q} \in X} R_{\mathfrak{q}}$, and let $h \colon B \to D$ and $k \colon C \to D$ be given by:

$$h(b) = (h_{\mathfrak{q}}(b + q_1))_{\mathfrak{q} \in X} \quad \text{and} \quad k(c) = (k_{\mathfrak{q}}(c + q_2))_{\mathfrak{q} \in X} ,$$

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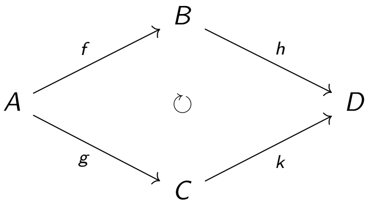
And finally the following diagram is commutative:



Two other cases

• Let $A = V_0 \times V_1$, where V_0, V_1 are two real closed valuation rings, then $\mathrm{T}_2 = \mathrm{Th}(A)$ is the theory of sc-regular projectable real closed rings with the first convexity property where the Boolean algebra of idempotents satisfies $\exists_{=4}$, i.e.: the formula saying that the structure has four elements.

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As the previous case, the theory F_2 has elimination of quantifiers in the language $\mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |_{\mathrm{loc}}\}$.

The main theorem

Let A be a reduced projectable and divisible-projectable f-ring satisfying the first convexity property. Then A admits quantifier elimination in $\mathcal{L}_1 = \mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |_{\mathrm{loc}}\}$ if and only if one of the following conditions holds:

(i) A is a real closed field,

(ii) A is a product of two real closed fields,

(iii) A is a von Neumann regular real closed ring without non-zero minimal idempotents,

(iv) A is a real closed valuation ring,

(v) A is a product of two real closed valuation rings,

(vi) A is a sc-regular real closed ring without non-zero minimal idempotents.

The idea

The idea of the proof is to interpret the language $\mathcal{L}_1 = \mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |_{\mathrm{loc}}\}$ in a previous language where I had proved the same result.

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The previous language is $\mathcal{L}_2 = \mathcal{L}_{\text{lor}} \cup \{\text{div}(\cdot,\cdot)\}$, where

$$\begin{aligned}\text{div}(x,y) = c \quad \longleftrightarrow \quad & c \preceq y \wedge \exists z \exists w \big(x = z + w \wedge z \perp w \wedge cy = z \wedge \\ & \forall w'(w' \neq 0 \wedge w' \perp (w - w') \to y \nmid w')\big).\end{aligned}$$

Real closed domains

A domain D is **real closed** if the following conditions are verified:

(i) $R = \mathrm{qf}(D)$ is a real closed field,

(ii) D is integrally closed (in R),

(iii) $\forall a \forall b (0 < a < b \rightarrow b \mid a^2)$ is true in D.

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Therefore $k_D = D/m_D$ is a subfield of $k_V = V/m_V$, and $\mathfrak{P}_D = (k_D, k_V)$ is a pair of real closed fields interpretable in the original structure D.

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Proposition: If two real closed domains D_1 and D_2 are elementary equivalent, then $\mathfrak{P}_{D_1} \equiv \mathfrak{P}_{D_2}$.

Real closed domains

Is the converse true ?

Real closed domains

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What is $\mathrm{tp}(x/A)$ in the $\mathrm{Th}(D)$? (in order to carry on a back and forth between ω_1-saturated structures).

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Thanks

Thank you !