# A converse to Cartan's Theorem B: The extension property for real analytic and Nash sets

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# 1. Introduction

## 1.1. Analytic case

Let $\Omega \subset \mathbb{R}^n$ be an open set. A subset $X \subset \Omega$ has the *analytic extension property* if each analytic function $f : X \to \mathbb{R}$ extends to an analytic function on $\Omega$.

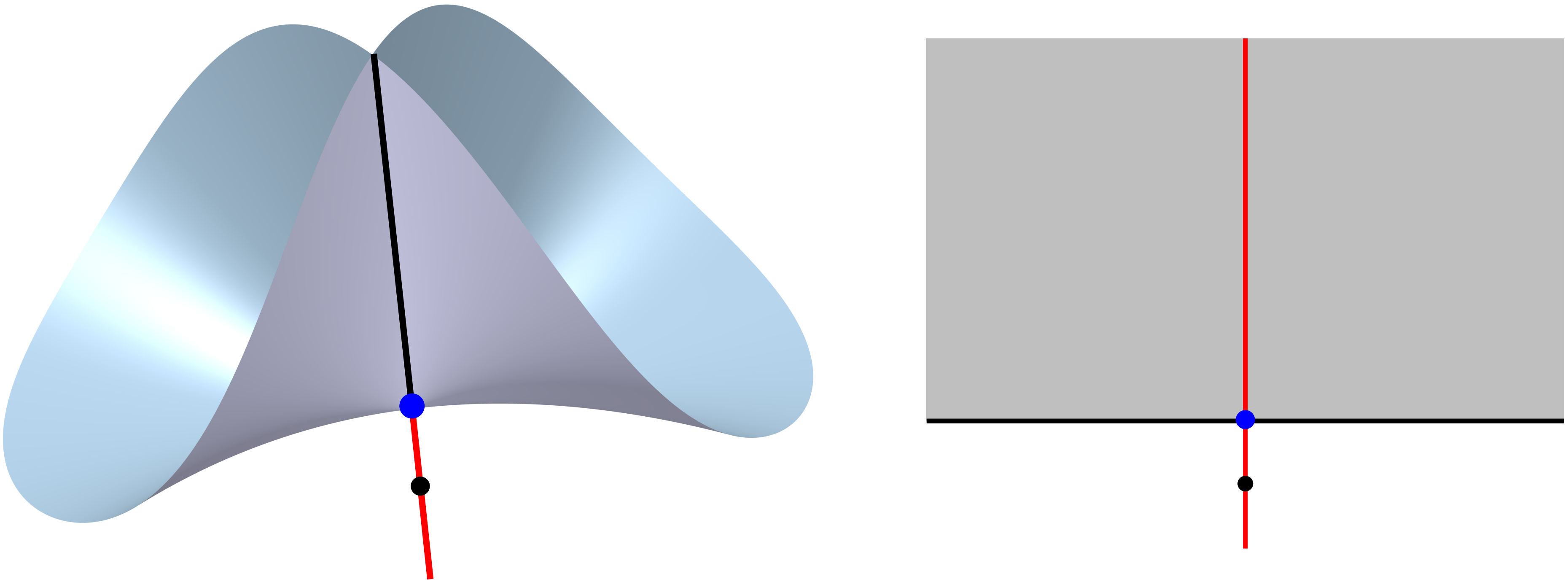
**Problem.** Which sets $X \subset \Omega$ have the analytic extension property?

**Necessary condition.** $X$ is the zero set of an analytic function on $\Omega \rightsquigarrow X$ is a $C$-analytic set.

**Example.** The necessary condition is not sufficient. Consider Whitney's umbrella $W := \{y^2 - zx^2 = 0\} \subset \mathbb{R}^3$ and

$$f : W \to \mathbb{R},\ (x, y, z) \mapsto \begin{cases} \frac{x}{z+1} & \text{if } (x, y, z) \neq (0, 0, -1), \\ 0 & \text{otherwise} \end{cases}$$

is analytic on $W$, but does not extend analytically to $\mathbb{R}^3$.



$$W := \{y^2 - zx^2 = 0\} \subset \mathbb{R}^3 \quad \text{and} \quad \xi := \frac{x}{z+1}$$

A sufficient condition is provided by coherence and Cartan's Theorem B.

**Coherence.** A $C$-analytic set $X$ is *coherent* if its local equations at each point $x \in X$ are generated by its global equations.

$$\mathcal{J}_{X,x} := \{f_x \in \mathcal{O}_{\mathbb{R}^n,x} : \ X_x \subset \mathcal{Z}(f_x)\} \quad \text{and} \quad \mathcal{I}(X) := \{f \in \mathcal{O}(\mathbb{R}^n) : \ X \subset \mathcal{Z}(f)\}$$

$$X \text{ is coherent } \iff \mathcal{J}_{X,x} = \mathcal{I}_{X,x} := \mathcal{I}(X)\mathcal{O}_{\mathbb{R}^n,x} \ \forall x \in X$$

**Cartan's Theorem B** (1957) $\Longrightarrow$ If $X \subset \Omega$ is a coherent $C$-analytic set, $X$ has the analytic extension property.

**Theorem (FG, 2025).** A set $X \subset \Omega$ has the analytic extension property $\iff$ $X$ is a coherent analytic set.
In addition, if $X$ is not coherent, there exist 'many' meromorphic functions on $\Omega$ that are analytic on $X$, but have no analytic extension to $\Omega$.

## 1.2. Nash case

**Semialgebraic set:** Boolean combination of sets defined by polynomial equalities and inequalities.

**Semialgebraic function:** Function with semialgebraic graph.

**Nash function on an open semialgebraic set $\Omega$:** Smooth + semialgebraic function on $\Omega \iff$ Analytic + semialgebraic function on $\Omega$.

**Nash manifold:** smooth manifold + semialgebraic set $\iff$ analytic manifold + semialgebraic set.

**Nash set:** Zero set of a Nash function on an open semialgebraic set $\iff$ $C$-analytic set + semialgebraic set.

**Local Nash function on a Nash set:** function on a Nash set that is locally at each of its points the restriction of a Nash function.

**Nash extension property:** Let $\Omega \subset \mathbb{R}^n$ be an open semialgebraic set. A subset $X \subset \Omega$ has the *Nash extension property* if each local Nash function $f : X \to \mathbb{R}$ extends to a Nash function defined on $\Omega$.

**Problem.** Which sets $X \subset \Omega$ have the Nash extension property?

**Necessary condition:** $X$ is a Nash set.

**Example.** The necessary condition is not sufficient. Consider Whitney's umbrella $W := \{y^2 - zx^2 = 0\} \subset \mathbb{R}^3$ and

$$f : W \to \mathbb{R},\ (x, y, z) \mapsto \begin{cases} \frac{x}{z+1} & \text{if } (x, y, z) \neq (0, 0, -1), \\ 0 & \text{otherwise} \end{cases}$$

is local Nash on $W$, but does not extend to $\mathbb{R}^3$ as a Nash function.

A sufficient condition is provided by coherence and Nash Theorem B.

**Coherence:** A Nash set $X$ is *coherent* if its local equations at each point $x \in X$ are generated by its global equations.

$$\mathcal{J}^{\bullet}_{X,x} := \{f_x \in \mathcal{N}_{\mathbb{R}^n,x} : \ X_x \subset \mathcal{Z}(f_x)\} \quad \text{and} \quad \mathcal{I}(X)^{\bullet} := \{f \in \mathcal{N}(\mathbb{R}^n) : \ X \subset \mathcal{Z}(f)\}$$

$$X \text{ is coherent } \iff \mathcal{J}^{\bullet}_{X,x} = \mathcal{I}^{\bullet}_{X,x} := \mathcal{I}^{\bullet}(X)\mathcal{N}_{\mathbb{R}^n,x} \ \forall x \in X$$

**Nash Theorem B** (Coste-Ruiz-Shiota, 2000) $\Longrightarrow$ If $X \subset \Omega$ is a coherent Nash set, $X$ has the Nash extension property.

**Theorem (FG, 2025).** $X \subset \Omega$ has the Nash extension property $\iff$ $X$ is a coherent Nash set.
In addition, if $X$ is not coherent, there exist 'many' meromorphic Nash functions on $\Omega$ that are locally Nash on $X$, but have no Nash extension to $\Omega$.

**Remark.** We 'semialgebraically' adapt the constructions done in the analytic case avoiding cohomology arguments.

Bad 'cohomological' behavior in the Nash case: $H^1(\mathbb{R}, \mathcal{N}_{\mathbb{R}}) \neq 0$ (Hubbard, 1972).

# 2. Analytic case

## 2.1. Coherence and Cartan's Theorems A & B

**Coherence.** A sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{R}^n}$-modules is *coherent* if:

(i) $\mathcal{F}$ is of finite type: $\forall x \in \mathbb{R}^n$ $\exists$ an open neighborhood $U \subset \mathbb{R}^n$ of $x$, $m \in \mathbb{N}^*$ and a surjective morphism $\mathcal{O}^m_{\mathbb{R}^n}|_U \to \mathcal{F}|_U$, and

(ii) the kernel of each homomorphism $\mathcal{O}^p_{\mathbb{R}^n}|_V \to \mathcal{F}|_V$ is of finite type for each $p \geq 1$ and each open subset $V$ of $\mathbb{R}^n$.

**Cartan's Theorems A and B.** describe the local-global behavior of coherent sheaves $\mathcal{F}$ of $\mathcal{O}_{\mathbb{R}^n}$-modules:

(A) *The stalks of a coherent sheaf $\mathcal{F}$ are spanned by the global sections.*

(B) *Each $p$-cohomology group of a coherent sheaf $\mathcal{F}$ is trivial for each $p > 0$.*

Let $X \subset \mathbb{R}^n$ be a $C$-analytic subset:

$\mathcal{C}^\omega_X := \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X$ is the *sheaf of analytic functions germs on* $X$

$\mathcal{O}_X := \mathcal{O}_{\mathbb{R}^n}/\mathcal{I}_X$ is the *sheaf of global analytic functions germs on* $X$.

$\mathcal{I}_X$ *is the biggest coherent* $\mathcal{O}_{\mathbb{R}^n}$*-sheaf of ideals with support* $X \Longrightarrow \mathcal{O}_{\mathbb{R}^n}/\mathcal{I}_X$ is coherent.

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{R}^n} \to \mathcal{O}_{\mathbb{R}^n}/\mathcal{I}_X \to 0 \quad \text{(exact sequence coherent sheaves)}$$

**Cartan's Theorem B** $\Longrightarrow H^1(\mathbb{R}^n, \mathcal{I}_X) = 0 \Longrightarrow$ The sequence

$$0 \to H^0(\mathbb{R}^n, \mathcal{I}_X) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/\mathcal{I}_X) \to 0$$

is exact.

$\mathcal{O}(X) := H^0(X, (\mathcal{O}_{\mathbb{R}^n}/\mathcal{I}_X)|_X)$ is the *ring of global analytic functions on* $X$

$\mathcal{C}^\omega(X) := H^0(X, (\mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X)|_X)$ is the *ring of analytic functions on* $X$.

**Analytic extension property:** A $C$-analytic set $X \subset \mathbb{R}^n$ has the *analytic extension property* if $\mathcal{O}(X) \to \mathcal{C}^\omega(X)$ is surjective.

## 2.2. 'Tails' and points of non-coherence of a $C$-analytic set

Let $X \subset \mathbb{R}^n$ be a $C$-analytic set.

### 2.2.1. Complexification of a $C$-analytic set.

Consider the coherent sheaf of $\mathcal{O}_{\mathbb{C}^n}$-ideals $\mathcal{I}_X \otimes_{\mathbb{R}} \mathbb{C}$ on $\mathbb{R}^n$. There exists an open neighborhood $\Omega \subset \mathbb{C}^n$ of $\mathbb{R}^n$ and a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{C}^n}$-ideals on $\Omega$ such that $\mathcal{F}|_{\mathbb{R}^n} = \mathcal{I}_X \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{I}(X)\mathcal{O}_{\mathbb{C}^n}|_{\mathbb{R}^n}$. A *complexification* $\widetilde{X}$ of $X$ is the support of $\mathcal{F}$.

### 2.2.2. Regular and singular points in the analytic setting

$x \in X$ is a *regular point of* $X$ if $\mathcal{O}(X)_{\mathfrak{m}_x}$ is a regular local ring. If one between $\mathcal{O}(X)_{\mathfrak{m}_x}$, $\mathcal{O}_{\mathbb{R}^n,x}/\mathcal{I}_{X,x}$, $\mathcal{O}_{\mathbb{C}^n,x}/(\mathcal{I}_{X,x} \otimes_{\mathbb{R}} \mathbb{C})$, $\mathcal{O}(\widetilde{X})_{\mathfrak{n}_x}$ is regular, all are regular.

$\operatorname{Reg}(X) = \operatorname{Reg}(\widetilde{X}) \cap X \rightsquigarrow$ *set of regular points* of $X$.

$\operatorname{Sing}(X) := X \setminus \operatorname{Reg}(X) = \operatorname{Sing}(\widetilde{X}) \cap X \rightsquigarrow$ *singular locus of* $X$ is a $C$-analytic set and $\dim(\operatorname{Sing}(X)) < \dim(X)$ (because $\dim(\operatorname{Sing}(\widetilde{X})) < \dim(\widetilde{X})$).

### 2.2.3. $C$-semianalytic sets

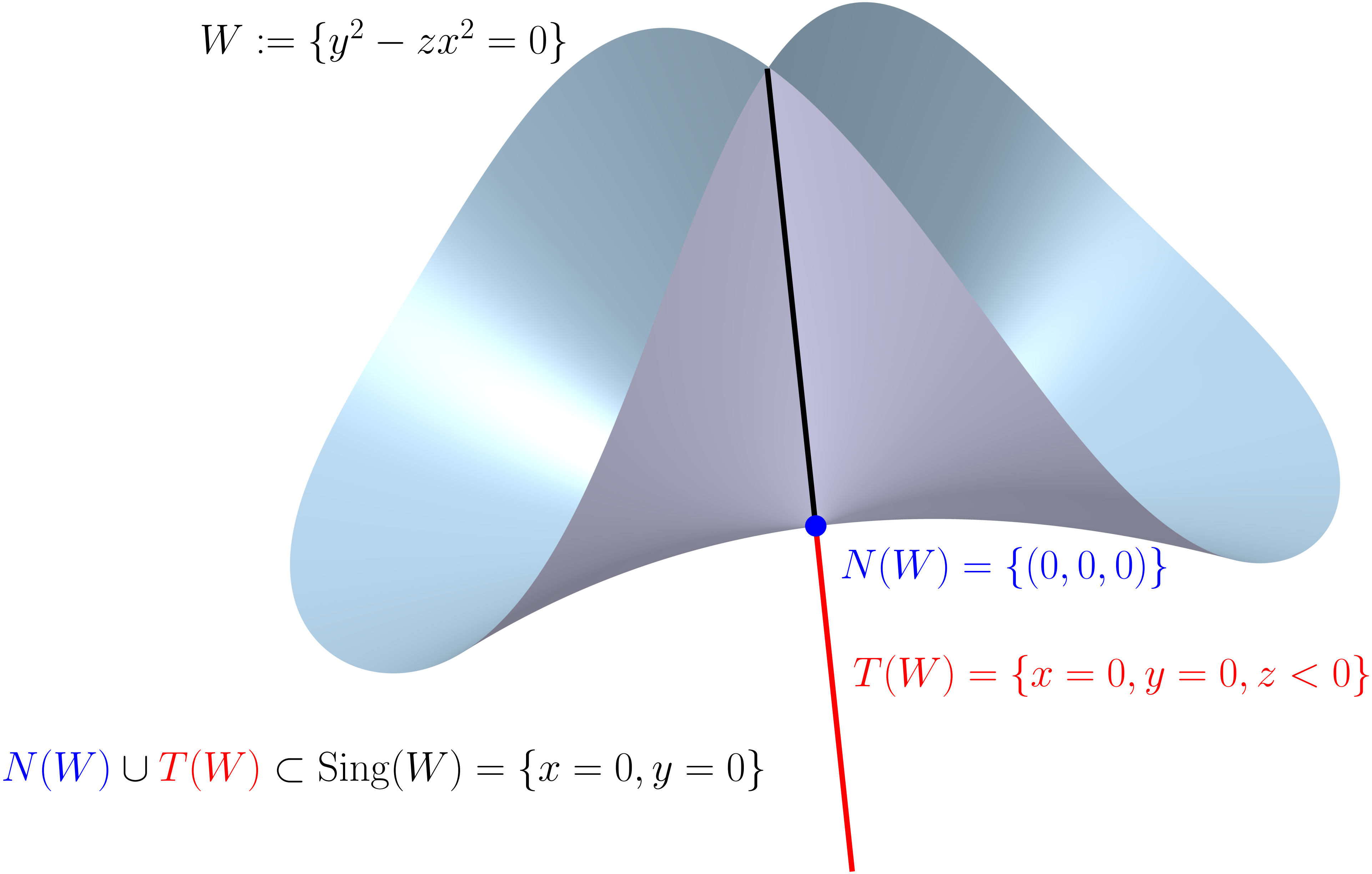
A $C$-semianalytic subset $S$ of $\mathbb{R}^n$ is a locally finite union of *basic $C$-semianalytic subsets* of $\mathbb{R}^n$ $\rightsquigarrow$ $\{f=0, g_1>0, \ldots, g_r>0\}$ where $r\geq 1$ and $f, g_i \in \mathcal{O}(\mathbb{R}^n)$.

### 2.2.4. Set of 'tails' of a $C$-analytic set

$T(X) := \{x \in X : \ \mathcal{J}_{X,x} \neq \mathcal{I}_{X,x}\}$ is the set of 'tails' of $X$ $\rightsquigarrow$ $T(X) \subset \operatorname{Sing}(X)$ is a $C$-semianalytic set of dimension $\dim(T(X)) < \dim(X)$.

### 2.2.5. Set of points of non-coherence.

The set $N(X)$ of points of non-coherence of $X$ is the set of points $x \in X$ such that $\mathcal{J}_X$ is not of finite type at $x$ (for each $x \in U \overset{\text{open}}{\subset} X$ the restricted sheaf $\mathcal{J}_X|_U$ is not of finite type) $\rightsquigarrow$ $N(X)$ is a closed $C$-semianalytic subset of $X$ of dimension $\leq \dim(X) - 2$.



$W := \{y^2 - zx^2 = 0\}$
$N(W) = \{(0,0,0)\}$
$T(W) = \{x = 0, y = 0, z < 0\}$
$N(W) \cup T(W) \subset \mathrm{Sing}(W) = \{x = 0, y = 0\}$

## 2.2.6. Properties of 'tails' and points of non-coherence

Let $X \subset \mathbb{R}^n$ be a $C$-analytic set.

**(1)** $\mathrm{Cl}(T(X)) = \mathrm{Cl}(T(X) \setminus N(X)) = T(X) \cup N(X)$.

**(2)** $X$ is coherent $\iff T(X) = \varnothing \iff N(X) = \varnothing$

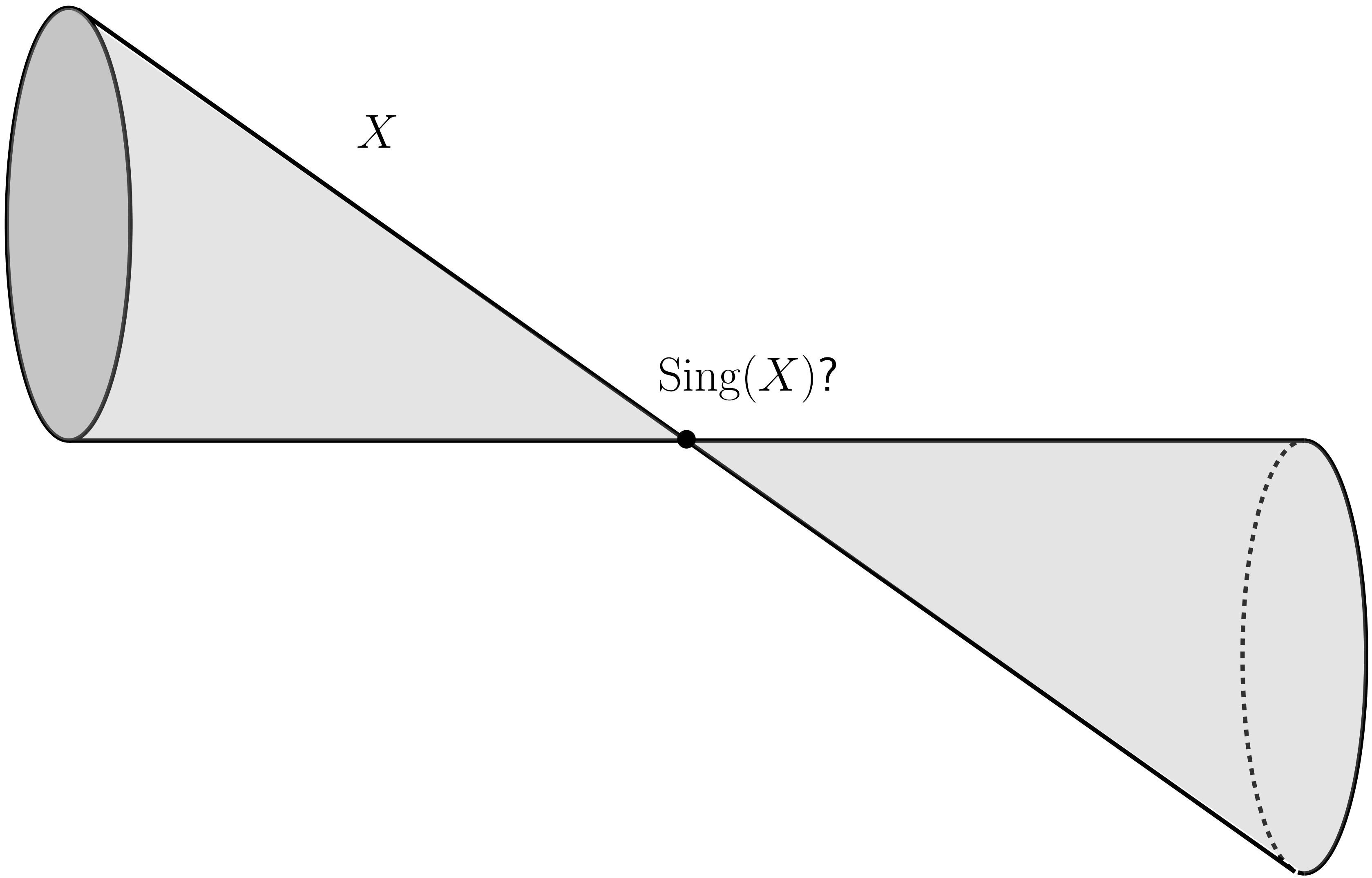
**(3)** If $S$ is a connected component of $\mathrm{Cl}(T(X))$, then $S \cap N(X) \neq \varnothing$.

**(4)** $\dim(N(X)_x) < \dim(T(X)_x) \leq \dim(\mathrm{Sing}(X)_x)$ for each $x \in N(X)$.

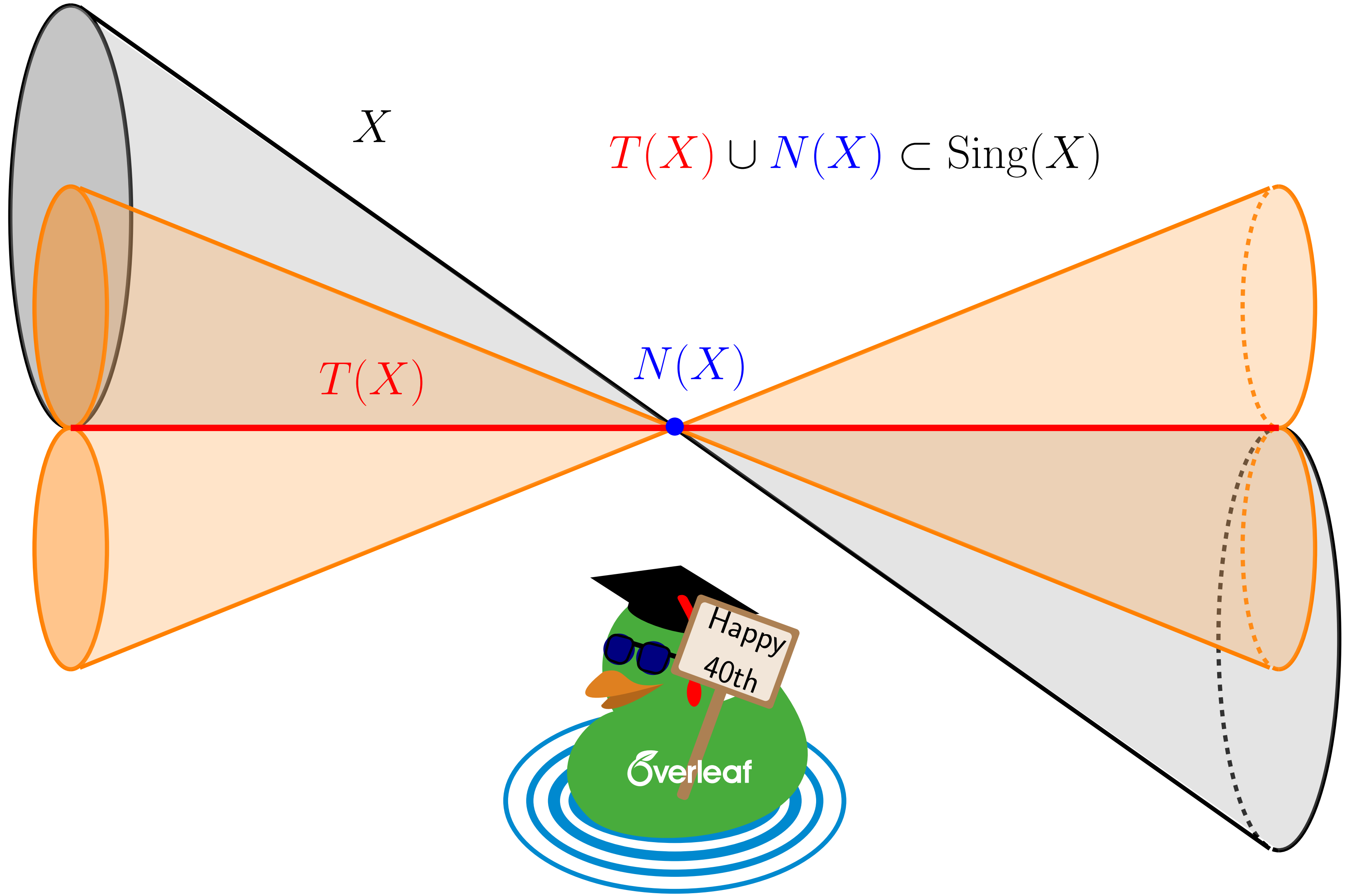
**(5)** $N(X) \cap T(X)$ may be non-empty, even if $X$ is $C$-irreducible.

**(6)** A general idea in Real Geometry is that non-coherence arises when the irreducible components of the objects are not pure dimensional.

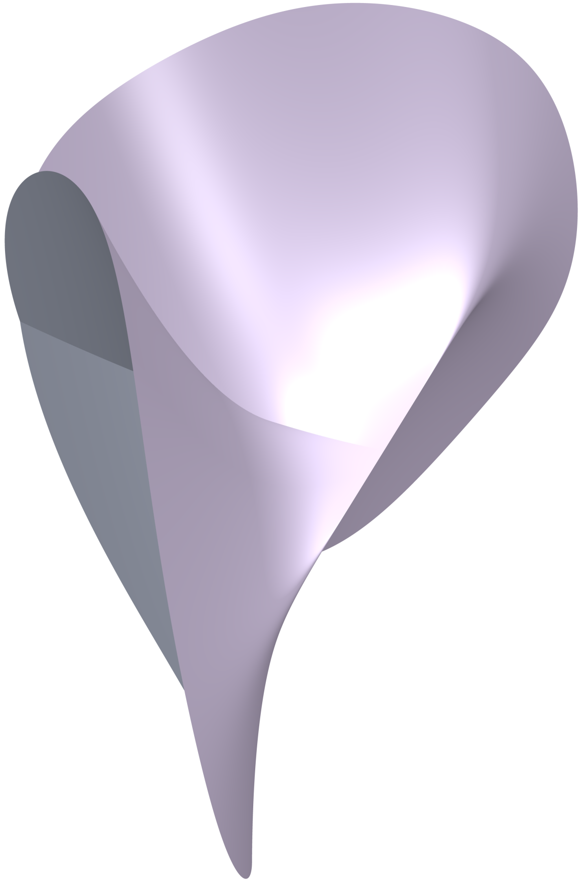
**(1), (2), (3)** with 'Real Vision Glasses'

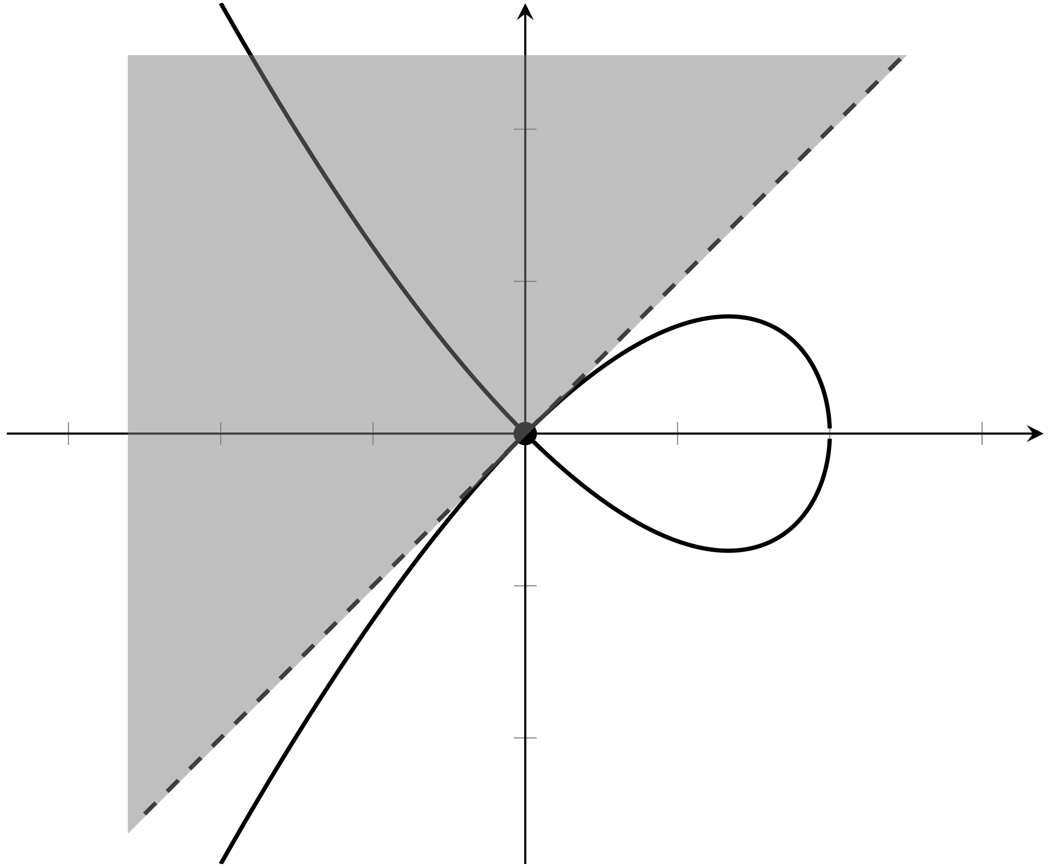


## **(1), (2), (3)** with 'Imaginary Vision Glasses'



**(5)**





- $X := \{(x^2 - y^2 - x^3)^2(y - x) - z^2 = 0\}$
- $T(X) = \{x^2 - y^2 - x^3 = 0, x - y > 0\} \cup \{(0,0,0)\}$ and $N(X) = \{(0,0,0)\}$
- $N(X) \cap T(X) = \{(0,0,0)\} \neq \varnothing$

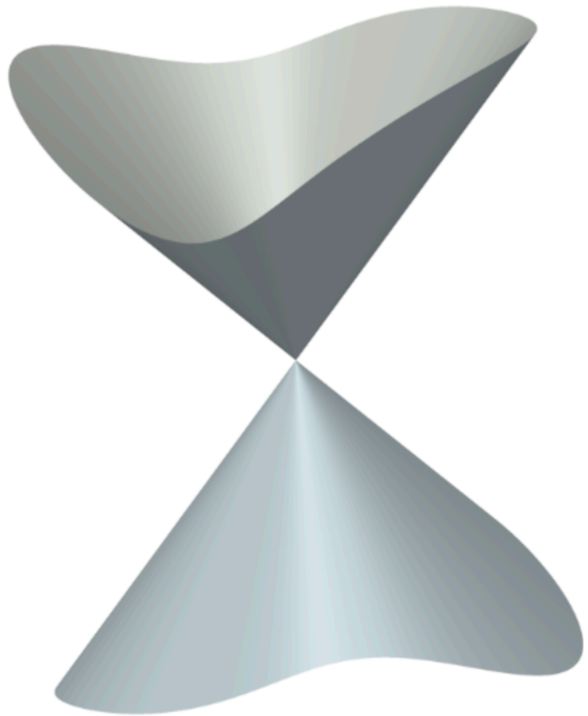
## 2.3. Examples of pure dimensional non-coherent $C$-analytic sets

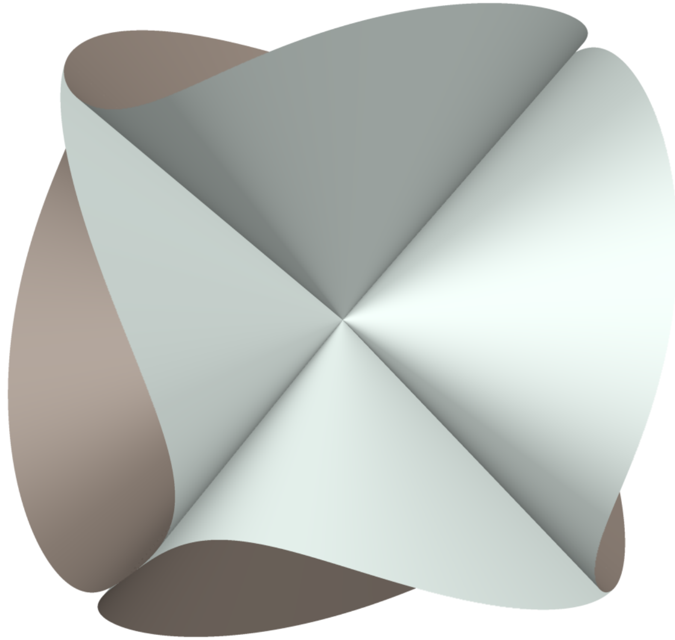
**(i) Galbiati-Hironaka:** $X := \{z(x+y)(x^2+y^2) - x^4 = 0\} \subset \mathbb{R}^3$

$\rightsquigarrow$ $N(X) = \{(0,0,0)\}$, $\operatorname{Sing}(X) = \{x = 0, y = 0\}$ and $T(X) = \operatorname{Sing}(X) \setminus N(X)$.

**(ii) Galbiati-Hironaka (modified):** $X := \{z^2(x+y)^2(x^2+y^2) - x^6 = 0\} \subset \mathbb{R}^3$

$\rightsquigarrow$ $N(X) = \{(0,0,0)\}$, $\operatorname{Sing}(X) = \{x = 0, yz = 0\}$ and $T(X) = \{x = 0, y = 0\} \setminus N(X)$.



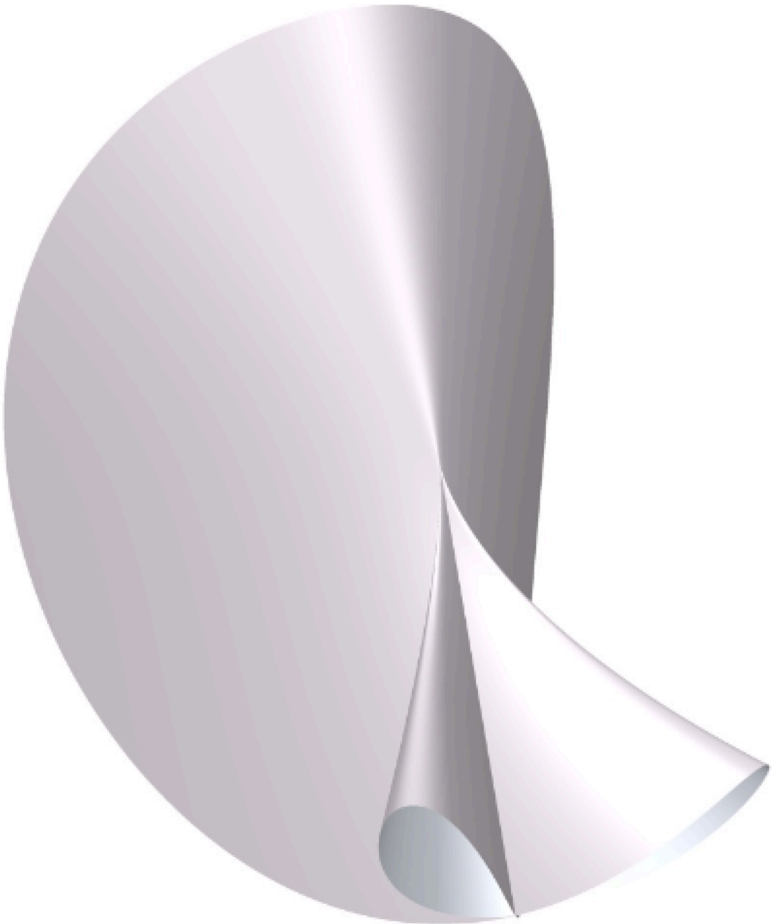


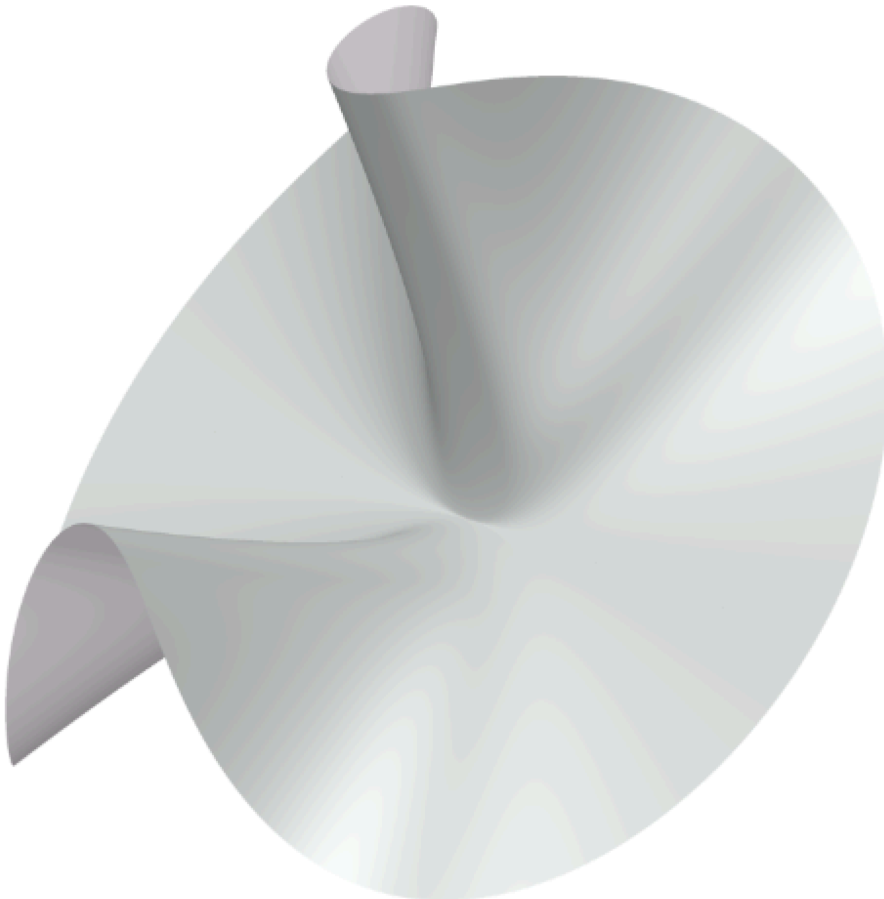
**(iii) Birdie non-coherent singularity:** $X := \{(x^2 + zy^2)x - y^4 = 0\} \subset \mathbb{R}^3$

$\rightsquigarrow$ $N(X) = \{(0,0,0)\}$, $\mathrm{Sing}(X) = \{x = 0, y = 0\}$ and $T(X) = \mathrm{Sing}(X) \cap \{z < 0\}$.

**(iv) Fake blanket:** $X := \{(x^2 + z^2y^2)x - y^4 = 0\} \subset \mathbb{R}^3$

$\rightsquigarrow$ $N(X) = \{(0,0,0)\}$, $\mathrm{Sing}(X) = \{x = 0, y = 0\}$ and $T(X) = \mathrm{Sing}(X) \setminus N(X)$.





## 2.4. Obstructing set of a meromorphic function

**(1)** Let $X$ be a $C$-analytic subset of $\mathbb{R}^n$, let $\zeta : X \to \mathbb{R}$ and $x \in X$ such that $\exists f_x, g_x \in \mathcal{O}_{\mathbb{R}^n,x}$ satisfying $\zeta_x = \frac{f_x}{g_x}$ and $g_x$ does not belong to a minimal prime of $\mathcal{J}_{X,x}$.

$$\frac{f_x}{g_x} = -a_x \in \mathcal{O}_{\mathbb{R}^n,x} \iff f_x + a_x g_x \in \mathcal{J}_{X,x} \iff f_x \in g_x \mathcal{O}_{\mathbb{R}^n,x} + \mathcal{J}_{X,x}.$$

**(2)** Let $\{X_i\}_{i\in I}$ be the $C$-analytic irreducible components of the $C$-analytic set $X$ and let $\zeta := \frac{f}{g} \in \mathcal{M}(X)$ such that $f, g \in \mathcal{O}(\mathbb{R}^n)$ and $g|_{X_i} \neq 0$ for each $i \in I$.

$$\zeta = \frac{f}{g} = -a|_X \text{ where } a \in \mathcal{O}(\mathbb{R}^n) \iff f \in g\mathcal{O}(\mathbb{R}^n) + \mathcal{I}(X)$$

$$\iff f_x \in g_x \mathcal{O}_{\mathbb{R}^n,x} + \mathcal{I}(X)\mathcal{O}_{\mathbb{R}^n,x} = g_x \mathcal{O}_{\mathbb{R}^n,x} + \mathcal{I}_{X,x} \ \forall x \in X$$

**Obstructing set of** $\zeta = \frac{f}{g} \in \mathcal{M}(X)$**:** $\mathtt{O}(\zeta) := \{x \in X : \ f_x \notin g_x \mathcal{O}_{\mathbb{R}^n,x} + \mathcal{I}_{X,x}\}$ (closed subset of $X$).

**Remark.** $\zeta \in \mathcal{M}(X)$ has an analytic extension to $\mathbb{R}^n \iff \mathtt{O}(\zeta) = \varnothing$.

## 2.5. Main Theorem

**Theorem.** Let $X \subset \mathbb{R}^n$ be a $C$-analytic set with $N(X) \neq \varnothing$. Let

- $Y \subset X$ be a $C$-analytic subset that contains no irreducible component of $X$ and meets $T(X)$,
- $U_0 \subset \mathbb{R}^n$ an open neighborhood of $Y$,
- $h \in H^0(U_0, \mathcal{J}_X)$ such that $h_y \in \mathcal{J}_{X,y} \setminus \mathcal{I}_{X,y}$ for each $y \in Y \cap T(X)$.

There exists $\zeta \in (\mathcal{M}(X) \cap \mathcal{C}^\omega(X)) \setminus \mathcal{O}(\mathbb{R}^n)$ such that $\mathsf{O}(\zeta) = Y \cap T(X)$.

**Remark.** We yield almost all possible obstructing sets (see §2.6.1).

## 2.6. Special global equations outside $N(X)$ ('Almost numerators')

**Theorem.** *Let $X \subset \mathbb{R}^n$ be a $C$-analytic set. Then there exists $h \in \mathcal{I}(X \setminus N(X))$ such that $\mathcal{Z}(h) = X \setminus N(X)$ and $h_x \in \mathcal{J}_{X,x} \setminus \mathcal{I}_{X,x}$ for each $x \in T(X) \setminus N(X)$.*

### 2.6.1. Winning equations around a $C$-analytic subset ('Numerators')

Let $X \subset \mathbb{R}^n$ be a $C$-analytic set and let $Y \subset X$ be a $C$-analytic subset.

**(1)** If $Y \subset X \setminus N(X) \subset U_0 := \mathbb{R}^n \setminus N(X)$, let $h \in \mathcal{O}(\mathbb{R}^n \setminus N(X))$ be such that $\mathcal{Z}(h) = X \setminus N(X)$ and $h_x \in \mathcal{J}_{X,x} \setminus \mathcal{I}_{X,x}$ $\forall x \in T(X) \setminus N(X)$. Then $\mathcal{Z}(h) = X \cap U_0$ and $h_y \in \mathcal{J}_{X,y} \setminus \mathcal{I}_{X,y}$ $\forall y \in Y \cap T(X)$.

**(2)** If $Y = \{y_k\}_{k\geq 1} \subset T(X)$ is a discrete set, $\forall k \geq 1$ $\exists y_k \subset V_k \overset{\text{open}}{\subset} \mathbb{R}^n$ and $h_k \in \mathcal{O}(V_k)$ such that $h_{k,y_k} \in \mathcal{J}_{X,y_k} \setminus \mathcal{I}_{X,y_k}$ and $\mathcal{Z}(h_{k,y_k}) = X_{y_k}$. Define $U_0 := \bigsqcup_{k\geq 1} V_k$,

$$h : U_0 \to \mathbb{R},\ x \mapsto h_k(x) \text{ if } x \in V_k.$$

$h \in \mathcal{O}(U_0)$ satisfies $\mathcal{Z}(h) = X \cap U_0$ and $h_y \in \mathcal{J}_{X,x} \setminus \mathcal{I}_{X,x}$ $\forall x \in Y \cap T(X)$.

No restriction with respect to the set $N(X)$.

## 2.7. Winning family of denominators

Let $Y \subset \mathbb{R}^n$ be a $C$-analytic subset of $\mathbb{R}^n$ and let $\widetilde{Y} \subset \Omega$ be an invariant Stein complexification of $Y$ closed in an open neighborhood $\Omega \subset \mathbb{C}^n$ of $\mathbb{R}^n$. Write $\widetilde{Y} = \mathcal{Z}(P_1, \dots, P_m)$ for some invariant $P_1, \dots, P_m \in \mathcal{O}(\Omega)$.

Define

$$\boxed{P_\lambda := \lambda_1 P_1^2 + \cdots + \lambda_m P_m^2 \in \mathcal{O}(\Omega)}$$

where $\lambda := (\lambda_1, \dots, \lambda_m) \in \mathfrak{Q}_m := \{\lambda_1 > 0, \dots, \lambda_m > 0\} \subset \mathbb{R}^m$. If $\Omega \subset \mathbb{C}^n$ is a contractible invariant open Stein neighborhood of $\mathbb{R}^n$, we can find a square-free invariant $P_\lambda^* \in \mathcal{O}(\Omega)$ such that $P_\lambda^* | P_\lambda$ and $\mathcal{Z}(P_\lambda^*) = \mathcal{Z}(P_\lambda)$.

**Examples.** (i) $Y := \{q\}$, $\widetilde{Y} := \{q\}$ and $P_i := \mathtt{x}_i - q_i$ for $i = 1, \dots, n$.

(ii) $Y := \{y_k\}_{k \geq 1}$ is a discrete subset of $T(X)$ and $\widetilde{Y} = Y$. If $Y = \mathbb{Z} \times \{(0, \overset{(n-1)}{\ldots}, 0)\}$, we take $P_1(x) := \sin(2\pi x_1)$, $P_2(x) = x_2$, $\dots$, $P_n(x) = x_n$.

## 2.8. Proof of the Main Theorem

STEP 1. *Initial preparation.* Consider the exact sequence of sheaves

$$0 \to \mathcal{J}_X \to \mathcal{O}_{\mathbb{R}^n} \to \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X \to 0$$

and the exact long corresponding sequence of global sections

$$0 \to H^0(\mathbb{R}^n, \mathcal{J}_X) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) \to H^1(\mathbb{R}^n, \mathcal{J}_X) \to H^1(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$$
$$\to \cdots \to H^p(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^p(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) \to H^{p+1}(\mathbb{R}^n, \mathcal{J}_X) \to H^{p+1}(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}).$$

**Cartan's theorem B** $\Longrightarrow H^p(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) = 0 \ \forall p \geq 1 \Longrightarrow$

$$0 \to H^0(\mathbb{R}^n, \mathcal{J}_X) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) \xrightarrow{\delta} H^1(\mathbb{R}^n, \mathcal{J}_X) \to 0.$$

$H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X)$ is surjective $\iff H^1(\mathbb{R}^n, \mathcal{J}_X) = 0$.

If $X$ is coherent $\overset{\text{(Cartan's B)}}{\Longrightarrow} H^1(\mathbb{R}^n, \mathcal{J}_X) = 0$ (because $\mathcal{J}_X = \mathcal{I}_X$).

**Purpose:** $X$ *is non-coherent* $\Longrightarrow H^1(\mathbb{R}^n, \mathcal{J}_X) \neq 0$.

### 2.8.1. Relations for $1$-cocycle and $1$-coboundary

Let $\mathfrak{U} := \{U_0, U_1\}$ be an open covering of $\mathbb{R}^n$. Let $\mathcal{F}$ be a sheaf on $\mathbb{R}^n$.

Pick $i, j, k \in \{0, 1\}$ and $f_{ij} \in \mathcal{F}(U_i \cap U_j)$, $f_{ik} \in \mathcal{F}(U_i \cap U_k)$, $f_{jk} \in \mathcal{F}(U_j \cap U_k)$.

$1$-**cocycle relations**: $f_{00} = 0$ on $U_0$, $f_{11} = 0$ on $U_1$ and $f_{10} = -f_{01}$ on $U_0 \cap U_1$.

$1$-**coboundary relations** for $f_0 \in \mathcal{F}(U_0)$ and $f_1 \in \mathcal{F}(U_1)$:

$$\begin{aligned} U_0 \cap U_1 &\rightsquigarrow f_{01} = f_1 - f_0, \\ U_1 \cap U_0 &\rightsquigarrow f_{10} = f_0 - f_1. \end{aligned}$$

A $1$-**cocycle**

$$(f_{00}, f_{01}, f_{10}, f_{11}) = (0, f_{01}, -f_{01}, 0) \in \mathcal{F}(U_0 \cap U_0) \times \mathcal{F}(U_0 \cap U_1) \times \mathcal{F}(U_1 \cap U_0) \times \mathcal{F}(U_1 \cap U_1)$$

is a $1$-**coboundary** $\iff \exists f_i \in \mathcal{F}(U_i)$ for $i \in \{0, 1\}$ such that $f_{01} = f_1 - f_0$.

STEP 2. *Construction of an open covering* $\mathfrak{U} := \{U_0, U_1\}$ *of* $X$ *and a* $1$*-cocycle* $f_{01} \in H^0(U_0 \cap U_1, \mathcal{J}_X)$ *that is not a* $1$*-coboundary.*

By hypothesis we have a $C$-analytic set $Y \subset X$, $Y \subset U_0 \overset{\text{open}}{\subset} \mathbb{R}^n$ and $h \in \mathcal{O}(U_0)$ such that $\mathcal{Z}(h) = X \cap U_0$ and $h_y \in \mathcal{J}_{X,y} \setminus \mathcal{I}_{X,y}$ $\forall y \in Y \cap T(X)$.

Let $\widetilde{Y} \subset \Omega \overset{\text{open}}{\subset} \mathbb{C}^n$ be a complexification of $Y$ and let $P_1, \dots, P_m \in \mathcal{O}(\Omega)$ be invariant such that $\widetilde{Y} = \mathcal{Z}(P_1, \dots, P_m)$.

$\mathfrak{U} := \{U_0, U_1 := \mathbb{R}^n \setminus Y\}$ open covering of $\mathbb{R}^n$. Define

$$P_\lambda(\mathbf{z}) := \lambda_1 P_1^2 + \cdots + \lambda_m P_m^2 \quad \text{and} \quad \boxed{f_{01,\lambda}(\mathbf{z}) := \frac{h(\mathbf{z})}{P_\lambda(\mathbf{z})} \in H^0(U_0 \cap U_1 = U_0 \setminus Y, \mathcal{J}_X)}$$

where $\lambda \in \mathfrak{Q}_m := \{\lambda_1 > 0, \dots, \lambda_m > 0\}$.

Let $f_0 \in H^0(U_0, \mathcal{J}_X)$ and $f_1 \in H^0(U_1, \mathcal{J}_X)$ be such that $f_{01,\lambda} = f_1 - f_0$ on $U_0 \cap U_1$.

$$g_\lambda := f_1 P_\lambda = h + f_0 P_\lambda \in \mathcal{O}(\mathbb{R}^n) \quad \text{and} \quad g_{k,\lambda}|_X = 0$$

If $G_\lambda$ is an analytic extension to $\Omega' \subset \Omega \Longrightarrow G_\lambda|_{\widetilde{X} \cap \Omega'} = 0$.

Let $V_0 \subset \mathbb{C}^n$ be an open neighborhood of $U_0$ such that $h, f_0$ extend to invariant holomorphic functions $H, F_0$ on $V_0$:

$$G_\lambda|_{V_0} = H + F_0 P_\lambda|_{V_0}.$$

We choose $\lambda_0 \in \mathfrak{Q}_m$ such that $\dim_{\mathbb{C}}((\widetilde{X}_y \cap \mathcal{Z}(P_{\lambda_0,y}) \setminus \mathcal{Z}(H_y)) \geq 1 \; \forall y \in Y \cap T(X)$.

$\forall y \in Y \cap T(X) \; \exists \beta^y : (-1,1) \to \widetilde{X}$ analytic curve such that $\beta^y(0) = y$ and $\beta^y((0,1)) \subset (\widetilde{X} \cap \mathcal{Z}(P_{\lambda_0})) \setminus \mathcal{Z}(H)$. Then

$$G_{\lambda_0} \circ \beta^y = 0, P_{\lambda_0} \circ \beta^y = 0, H \circ \beta^y \neq 0$$
$$\rightsquigarrow 0 = G_{\lambda_0} \circ \beta^y = H \circ \beta^y + (F_0 \circ \beta^y)(P_{\lambda_0} \circ \beta^y) = H \circ \beta^y \quad !!!!!$$

**Conclusion:** $f_{01,\lambda_0} \in H^0(U_0 \cap U_1, \mathcal{J}_X)$ is $1$-cocycle that it is not a $1$-coboundary.

The obstruction concentrates at all the points of $Y \cap T(X)$.

STEP 3. *Construction of* $\zeta \in \mathcal{C}^\omega(X) \setminus \mathcal{O}(X)$. (We use the 1-cocycle)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(U_0, \mathcal{J}_X) \times H^0(U_1, \mathcal{J}_X) & \longrightarrow & \mathcal{O}_{\mathbb{R}^n}(U_0) \times \mathcal{O}_{\mathbb{R}^n}(U_1) & \longrightarrow & H^0(U_0, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) \times H^0(U_1, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(U_0 \cap U_1, \mathcal{J}_X) & \longrightarrow & \mathcal{O}_{\mathbb{R}^n}(U_0 \cap U_1) & \longrightarrow & H^0(U_0 \cap U_1, \mathcal{O}_{\mathbb{R}^n}/\mathcal{J}_X) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccc}
(f_0, f_1) & \longmapsto & (f_0, f_1) \\
\downarrow & & \downarrow \\
f_1 - f_0 & \longmapsto & f_1 - f_0
\end{array}$$

$$\begin{array}{ccc}
(q_0, q_1) & \longmapsto & ([q_0], [q_1]) \\
\downarrow & & \downarrow \\
q_1 - q_0 & \longmapsto & [q_1 - q_0]
\end{array}$$

We seek $(q_0, q_1) \in \mathcal{O}_{\mathbb{R}^n}(U_0) \times \mathcal{O}_{\mathbb{R}^n}(U_1)$ such that

$$\frac{h}{P_{\lambda_0}} = f_{01,\lambda_0} = q_1 - q_0 = \delta((q_0, q_1))$$

$$\rightsquigarrow \;\; h_{\lambda_0} := q_1 P_{\lambda_0} = h + q_0 P_{\lambda_0} \in \mathcal{O}(\mathbb{R}^n) \quad \& \quad q_0 := \frac{h_{\lambda_0} - h}{P_{\lambda_0}} \in \mathcal{O}(U_0).$$

Consider the exact sequence of coherent $\mathcal{O}_{\mathbb{R}^n}$-sheaves

$$0 \to P_{\lambda_0}\mathcal{O}_{\mathbb{R}^n} \to \mathcal{O}_{\mathbb{R}^n} \to \mathcal{O}_{\mathbb{R}^n}/P_{\lambda_0}\mathcal{O}_{\mathbb{R}^n} \to 0.$$

Cartan's Theorem B $\Longrightarrow$

$$0 \to H^0(\mathbb{R}^n, P_{\lambda_0}\mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}) \to H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/P_{\lambda_0}\mathcal{O}_{\mathbb{R}^n}) \to 0 \quad \text{(exact)}$$

As $h \in H^0(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n}/P_{\lambda_0}\mathcal{O}_{\mathbb{R}^n})$, $\exists h_{\lambda_0} \in \mathcal{O}(\mathbb{R}^n)$ such that $h_{\lambda_0} - h$ is a multiple of $P_{\lambda_0}$.

$$\begin{cases} q_0 := \frac{h_{\lambda_0}-h}{P_{\lambda_0}} \in \mathcal{O}(U_0), \\ q_1 := \frac{h_{\lambda_0}}{P_{\lambda_0}} \in \mathcal{O}(U_1) \qquad \text{(because } \mathcal{Z}(P_{\lambda_0}) = Y) \end{cases} \quad \& \quad \delta((q_0, q_1)) = q_1 - q_0 = \frac{h}{P_{\lambda_0}} = f_{01}$$

Sought function.

$$q_{\lambda_0} : X \to \mathbb{R},\ z \mapsto \begin{cases} q_0(z) & \text{if } z \in X \cap U_0 = X \setminus N(X), \\ q_1(z) & \text{if } z \in X \cap U_1 = X \setminus Y \end{cases} \quad \& \quad \boxed{\xi_{\lambda_0} = \frac{h_{\lambda_0}}{P_{\lambda_0}} \in \mathcal{M}(X)}$$

Step 4. Obstruction set. $\mathsf{O}(\xi_{\lambda_0}) = Y \cap T(X)$.

# 3. Semialgebraic case

Let $S \subset \mathbb{R}^n$ be a semialgebraic set.

## 3.1. Differentiable semialgebraic functions

$f : S \to \mathbb{R}$ is a $\mathcal{S}^p$ *function (semialgebraic* $+$ $\mathcal{C}^p$ *function)* if $\exists S \subset U \overset{\text{open s.a.}}{\subset} \mathbb{R}^n$ and a $\mathcal{S}^p$-function (semialgebraic $+$ $\mathcal{C}^p$ function) $F : U \to \mathbb{R}$ such that $F|_S = f$.

## 3.2. Smooth semialgebraic functions and Nash functions

**Classical result:** smooth $+$ semialgebraic function on a Nash manifold $S \subset \mathbb{R}^n$ $\iff$ analytic $+$ algebraic function on $S$.

**Question:** What happens in the general semialgebraic setting?

The *ring of smooth semialgebraic functions on* $S$ is

$$\mathcal{S}^{(\infty)}(S) := \bigcap_{p\geq 0} \mathcal{S}^p(S)$$

where $\mathcal{S}^p(S)$ is the ring of $\mathcal{S}^p$ functions on $S$.

**Definition.** $\mathcal{N}(S) := H^0(S, (\mathcal{N}_{\mathbb{R}^n})|_S) = \varinjlim \mathcal{N}(V)|_S$ *where* $V \subset \mathbb{R}^n$ *covers the open semialgebraic neighborhoods of* $S$*. We have* $\mathcal{N}(S) \subset \mathcal{S}^{(\infty)}(S)$*.*

**Problem.** For which semialgebraic sets $S \subset \mathbb{R}^n$ do we have $\mathcal{N}(S) = \mathcal{S}^{(\infty)}(S)$?

## 3.3. Nash sets

Let $X \subset \mathbb{R}^n$ be a Nash set.

**(1)** $\mathcal{S}^{(\infty)}(X) = \mathcal{C}^{\mathcal{N}}(X)$ and $\mathcal{N}(X) = H^0(X, \mathcal{N}_{\mathbb{R}^n}/\mathcal{I}_X^\bullet)$.

**(2)** $\mathcal{S}^{(\infty)}(X) = \mathcal{C}^{\mathcal{N}}(X) = \mathcal{N}(X) \iff X$ is coherent ($\mathcal{J}_{X,x}^\bullet = \mathcal{I}_{X,x}^\bullet \ \forall x \in X$).

## 3.4. Semialgebraic sets

Let $S \subset \mathbb{R}^n$ be a semialgebraic set and define

$$\mathcal{J}^{\bullet}_{S,x} := \{f_x \in \mathcal{N}_{\mathbb{R}^n,x} : \ S_x \subset \mathcal{Z}(f_x)\} \quad \text{and} \quad \mathcal{C}^{\mathcal{N}}(S) := H^0(S, \mathcal{N}_{\mathbb{R}^n}/\mathcal{J}^{\bullet}_S).$$

- $\mathcal{S}^{(\infty)}(S) = \mathcal{C}^{\mathcal{N}}(S)$.
- $\mathcal{N}(S) = H^0(S, (\mathcal{N}_{\mathbb{R}^n}/\mathcal{I}^{\bullet}_X)|_S) = \varinjlim \mathcal{N}(V)/\mathcal{I}^{\bullet}(X)\mathcal{N}(V)$ where $V \subset \mathbb{R}^n$ covers the open semialgebraic neighborhoods of $S$ and $X$ is a suitable Nash closure of $S$.

### 3.4.1. Smooth semialgebraic functions versus Nash functions

We have $\mathcal{I}^{\bullet}_{X,x} \subset \mathcal{J}^{\bullet}_{S,x}$ for each $x \in S$. Define $T(S) := \{x \in S : \ \mathcal{I}^{\bullet}_{X,x} \neq \mathcal{J}^{\bullet}_{S,x}\}$.

$$\boxed{\mathcal{N}(S) = \mathcal{S}^{(\infty)}(S) \iff T(S) = \varnothing}$$

**Example.** $S := \{x^2 - zy^2 = 0, z \geq 0\}$ *(Whitney's umbrella with the handle erased) has* $T(S) = \varnothing$*.*