Split hyperplane sections on polarized $K3$-surfaces

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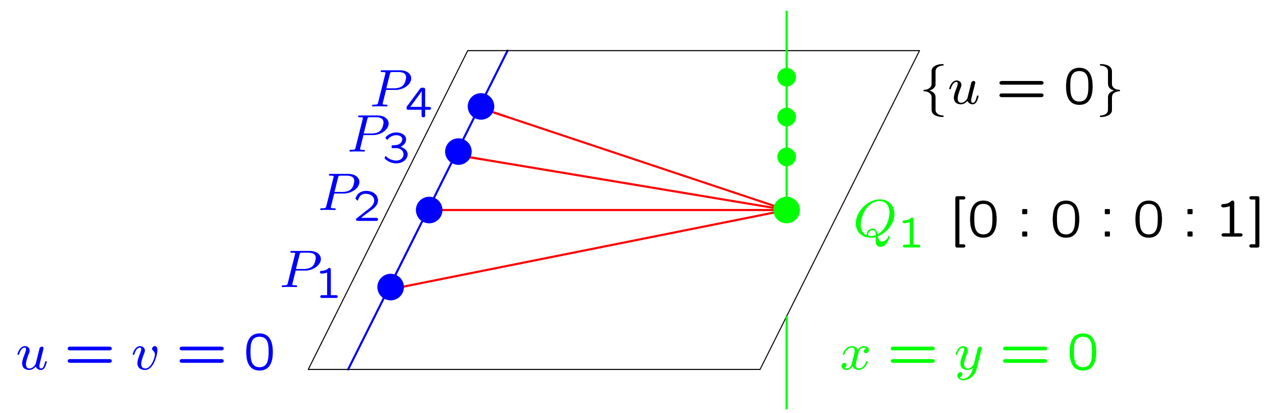
An old example [F. Schur, 1882]

Consider the quartic $X = \{\varphi(x,y) = \psi(u,v)\}$, $\deg\varphi = \deg\psi = 4$.
(Here, $[x : y : u : v]$ are homogeneous coordinates in $\mathbb{P}^3$.)

$$\left.\begin{array}{r} P_1,\ldots,P_4 = \text{roots of } \varphi \text{ on } \{u=v=0\} \\ Q_1,\ldots,Q_4 = \text{roots of } \psi \text{ on } \{x=y=0\} \end{array}\right\} \Rightarrow (P_i, Q_j) \subset X.$$

Thus, we get 16 lines in X.



Wren, 1669; Shukhov, 1880's.

Cayley, 1849; Salmon, 1862

→ **An old example** [F. Schur, 1882]

What if $\varphi = \psi$? For $k = 0, 1, 2, 3$,

$$\left.\begin{array}{l} u = i^k x \\ v = i^k y \end{array}\right\} \Rightarrow \varphi(x,y) = \varphi(i^k x, i^k y) = i^{4k}\varphi(x,y),$$

i.e., we get 4 more lines. For each Möbius transformation $[u:v] \mapsto [u':v']$ preserving φ, we get 4 more. Thus:

$\varphi \neq \psi$:	16 lines,
$\lvert\mathrm{Aut}\,\varphi\rvert = 4$ (generic) :	32 lines,
$\lvert\mathrm{Aut}\,\varphi\rvert = 8$ $(x^4 + y^4)$, *Fermat* :	48 lines,
$\lvert\mathrm{Aut}\,\varphi\rvert = 12$ $(x^4 + xy^3)$, *Schur* :	**64** lines.

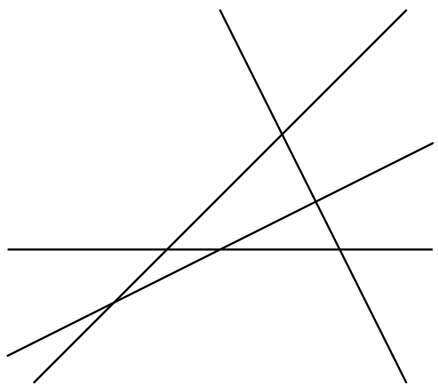
Remark 1 *A generic quartic has no lines:*

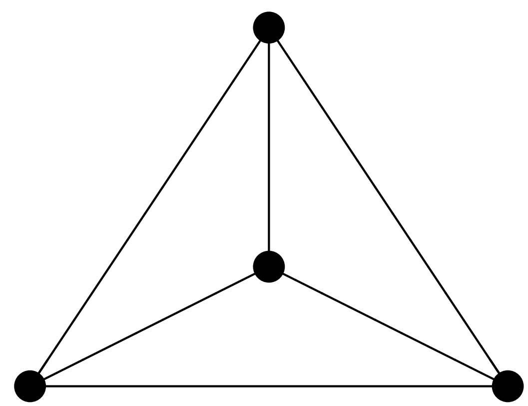
- codim$\{$*quartics* $\supset$ *a fixed line*$\} = 5$ (*five coeffs vanish*);
- $\dim \mathrm{Gr}(4,2) = 4 < 5$.

→ **An old example** [F. Schur, 1882]

Theorem 2 [Segre, 1943] *The maximal number of lines on a smooth quartic is* 64*.*

Proof: Pick a split plane section l_1, l_2, l_3, l_4 as in the example:





$K(4)$

Each line l_i intersects $n_i \leqslant 18$ other lines;
each other line intersects one of l_i. Hence, the number is

$$(n_1 - 3) + \ldots + (n_4 - 3) + 4 \leqslant 4(18 - 3) + 4 = 64. \qquad \square$$

$\rightarrow$ **An old example** [F. Schur, 1882]

Remark 3 *There are a few problems*:

- $n_i \leqslant 18$ *is not correct*: *in fact*, $n_i \leqslant 20$, *but these are rare* [Rams–Schütt, 2015; D.–Itenberg–Sertöz, 2016];
- *does there always exist a split section? No!* [loc. cit.]
  If not, at most 48 (39??) *lines* [D.–Rams, 2024].

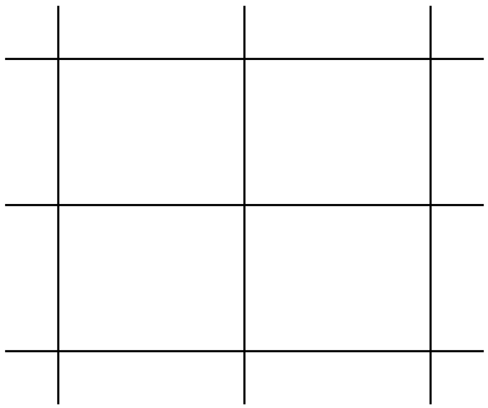
Numbers of lines are known [loc. cit.; D., 2019, 2022]:
tend to decrease, oscillate $\leqslant$ 24 as $\hbar^2 \to \infty$

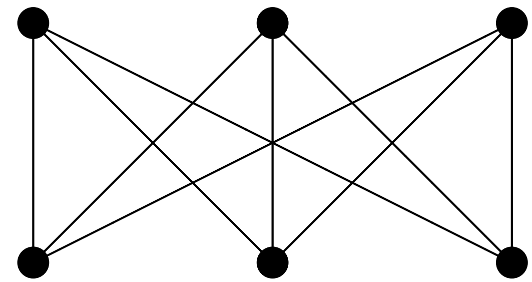
$\hbar^2$	2	4	6	8	10	12	14	16	18	20	28	else
M	144	64	42	36	30	36	30	32	25	25	28	$\leqslant 24$
$\bar{M}$	130	48	35	30	28	28	26	24	24	24	24	< 24

(We know all configurations with $> \bar{M}(\hbar)$ lines.)

Humbert sextics [D.–Dolgachev–Kōndo, 2025]

A *sextic $K3$-surface*: $X = Q_2 \cap Q_3 \subset \mathbb{P}^4$.
Hyperplane section: in $X \cap H \subset Q_2 \cap H = \mathbb{P}^1 \times \mathbb{P}^1$, deg $= (3,3)$.





$K(3,3)$

Humbert line complex: cut by a cubic hypersurface on the image $\mathrm{Gr}(4,2) \subset \mathbb{P}^5$ under the Plücker embedding.
Humbert sextic $K3$-surface X: a generic hyperplane section.
In $\mathrm{Fn}\, X$, there are 16 fragments like this; play a major rôle.

Fano graph

Problem 4 [Dolgachev, 2025] *What are the maximal numbers?*

- *On a smooth quartic $X \subset \mathbb{P}^3$, how many $\hbar$-fragments $K(4)$?*
- *On a smooth $K3$-sextic $X \subset \mathbb{P}^4$, how many $K(3,3)$'s?*

$K3$-surfaces

$K3$-surface X: compact surface/$\mathbb{C}$ with $\pi_1(X) = 0$, $K_X = 0$.
A class in *Enriques–Kodaira classification* ($\approx$ elliptic curves).
Unique minimal model ($\kappa = 0$) $\Rightarrow$ usually *minimal* and *smooth*.
A single deformation family $\Rightarrow$ topology is known.
Most non-algebraic; algebraic = countable $\bigcup$ of hypersurfaces.
Best known algebraic examples:

- deg = 2: double planes $X \xrightarrow{\times 2} \mathbb{P}^2 \supset C_6$;
- deg = 4: quartics $X_4 \subset \mathbb{P}^3$;
- deg = 6: sextics $Q_2 \cap Q_3 \in \mathbb{P}^4$;
- deg = 8: octics; mostly *triquadrics* $Q_2' \cap Q_2'' \cap Q_2''' \subset \mathbb{P}^5$.

In general, a *$K3$-surface of degree $h^2 = 2d$* (*$2d$-polarized*):

$$X \longrightarrow \mathbb{P}^{d+1}.$$

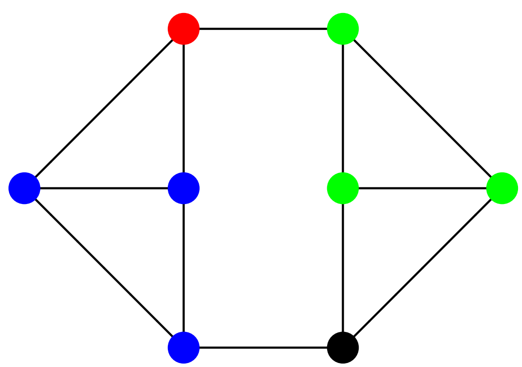
No longer complete intersections (for $h^2 \geqslant 10$).
Occasionally can allow $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$ singularities; here, all are smooth.

The magic of $K3$-surfaces

In a nutshell, given a graph Γ, there is a script [D.–Rams, 2025] that tells us if Γ is (a subgraph of) the *Fano graph* $\mathrm{Fn}\,X$ of a (smooth) $2d$-polarized $K3$-surface $X \subset \mathbb{P}^{d+1}$.

- [Pjateckiĭ-Šapiro–Šafarevič, 1971; Kulikov, 1977];
- [Riemann–Roch; Hodge; Saint-Donat, 1974];
- [Nikulin, 1979; Vinberg, 1972];
- [Beauville, Dolgachev, Huybrechts, . . .].

For example:



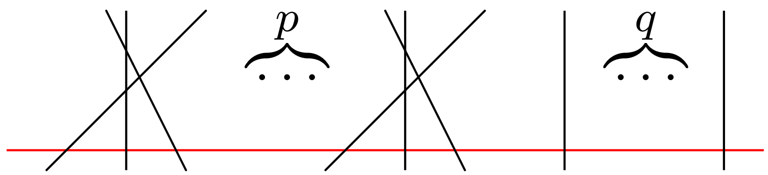
$\triangle := \sum \bullet, \quad \triangle := \sum \bullet.$

$$\triangle^2 = \triangle^2 = \triangle \cdot \triangle = 0 \quad \text{but} \quad v \cdot \triangle = 2 \neq 1 = v \cdot \triangle.$$

Quartics [D., 202?]

Theorem 5 *A quartic* $X \in \mathbb{P}^3$ *has* $\leqslant 72$ $K(4)$*-fragments.*

Proof: Consider a *bouquet of* $\hbar$*-fragments* at a line l:



$p =$	0	1	2	3	4	5	6
$q \leqslant$	12	10	9	7	6	3	2

(In particular, this implies val $l \leqslant 20$.)

Thus,

$$\#K(4) \leqslant \frac{6}{4}|\mathrm{Fn}\,X| \leqslant 72 \quad \text{if} \quad |\mathrm{Fn}\,X| \leqslant 48.$$

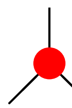
Configurations with >48 lines are known [D.–Rams, 2024]. □

Remark 6 *Conjecturally, the number of* $K(4)$*-fragments is* $\leqslant 48$*, with a few explicit exceptions.*
The only quartic with 72 fragments is Schur's:
the known champion in quite a few similar problems.

Sextics [D., 202?]

Properties of *$\hbar$-fragments* $\Sigma \subset \mathrm{Fn}\, X$ s.t. $\sum_{v\in\Sigma} v = \hbar$.
(All is happening in $NS(X) \approx (\mathbb{Z}\Gamma + \mathbb{Z}\hbar)/\text{radical}$.)

1. An $\hbar$-fragment Σ is a 3-regular (*aka cubic*) graph:



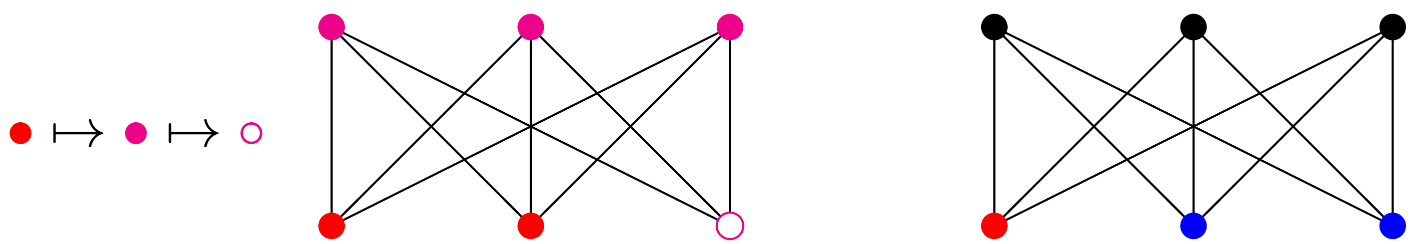
$$-2 + \operatorname{val} v = \hbar \cdot v = 1.$$

2. One has $|\Sigma| = \hbar^2$.
   (Automatically $|\Sigma| \geqslant \hbar^2$ as $\sum v$ is the *intrinsic polarization*.)

3. Any $u \in \mathrm{Fn}\, X \smallsetminus \Sigma$ is adjacent to **exactly one** $v \in \Sigma$.

4. $\triangle := \Sigma_1 \cap \Sigma_2$ is a *perfect subset* of Σ_i.
   Adjacency of $\Sigma_1 \smallsetminus \triangle$ and $\Sigma_2 \smallsetminus \triangle$ is a bijection of
   the *perfect complements* $\Sigma_1 \mathbin{\smallsetminus\!\!\smallsetminus} \triangle \cong \Sigma_2 \mathbin{\smallsetminus\!\!\smallsetminus} \triangle$ (as sets):

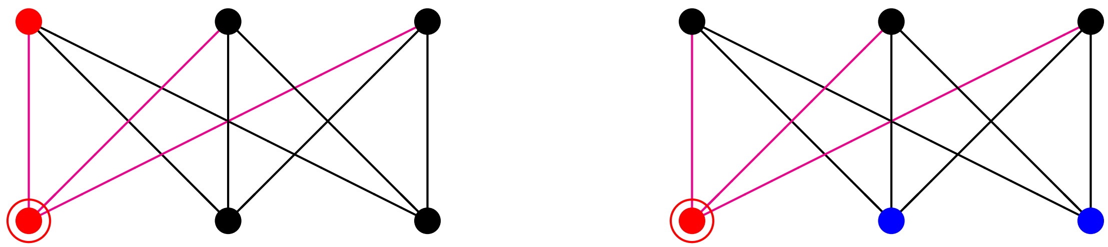
not graph!



→ **Sextics** [D., 202?]

Theorem 7 *A sextic* $X \in \mathbb{P}^4$ *has* $\leqslant 40$ $K(3,3)$*-fragments.*

Proof: there are but two proper perfect subsets of $K(3,3)$:



The star of a line in $\operatorname{Fn} X$ is $a\mathbf{A}_1 \oplus b\mathbf{A}_2$, $a \leqslant 9$, $b \leqslant 1$.
Hence, a bouquet of $K(3,3)$-fragments is (almost) determined by their "germs", *i.e.*, a collection $\mathcal{S}$ of 3-elements subsets

$$s \subset \mathfrak{S} := \{1, \ldots, 9\} \quad \text{s.t.} \quad |r \mathbin{\triangle} s| \in \{0, 4, 6\} \quad \forall\, r, s \in \mathcal{S}.$$

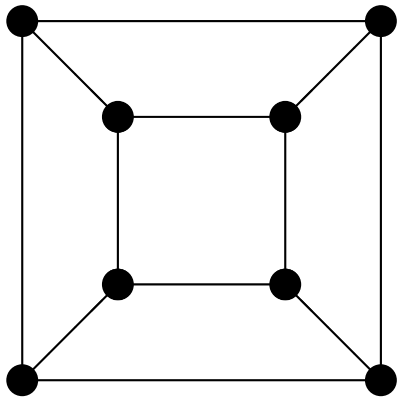
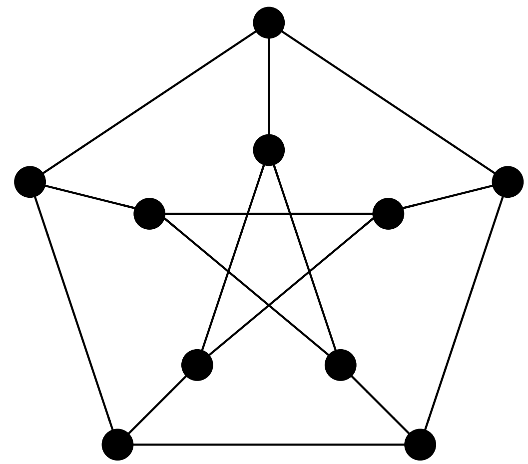
One has $|\mathcal{S}| \leqslant 12$; the two sets with $|\mathcal{S}| = 11, 12$ are ruled out. Thus,

$$\#K(3,3) \leqslant \frac{10}{6}|\operatorname{Fn} X| \leqslant 58 \quad \text{if} \quad |\operatorname{Fn} X| \leqslant 35.$$

Alas, we get no proof!! □

Degrees 8 and 10

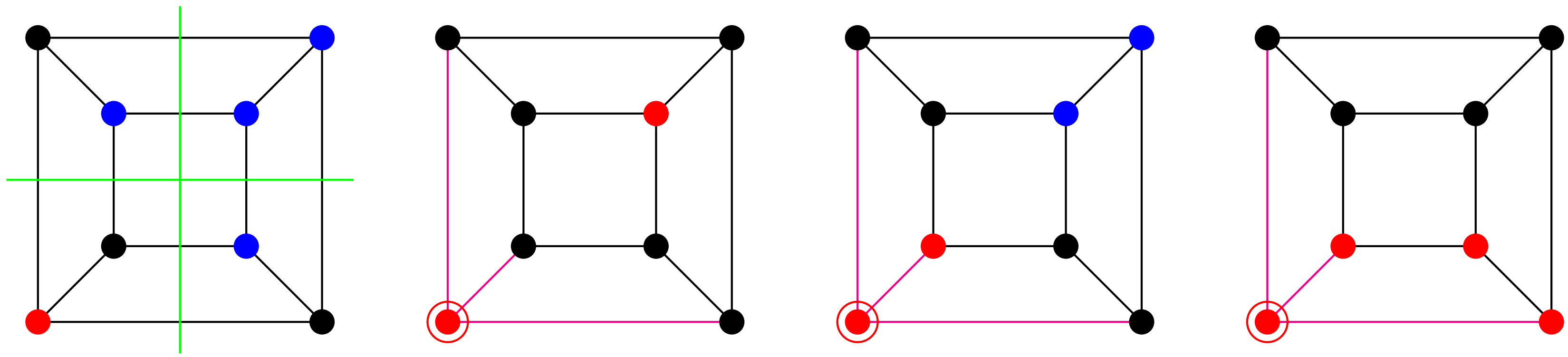
Suggested by wild guessing and my sense of beauty:

$\hbar^2 = 8$:  $\hbar^2 = 10$: 

Theorem 8 [D., 202?] *The sharp upper bounds are*:

- *at most* 80 *cube fragments* ($\hbar^2 = 8$);
- *at most* 16 *Petersen fragments* ($\hbar^2 = 10$).

$\rightarrow$ **Degrees** 8 **and** 10



The first one is ruled out by our "magic."

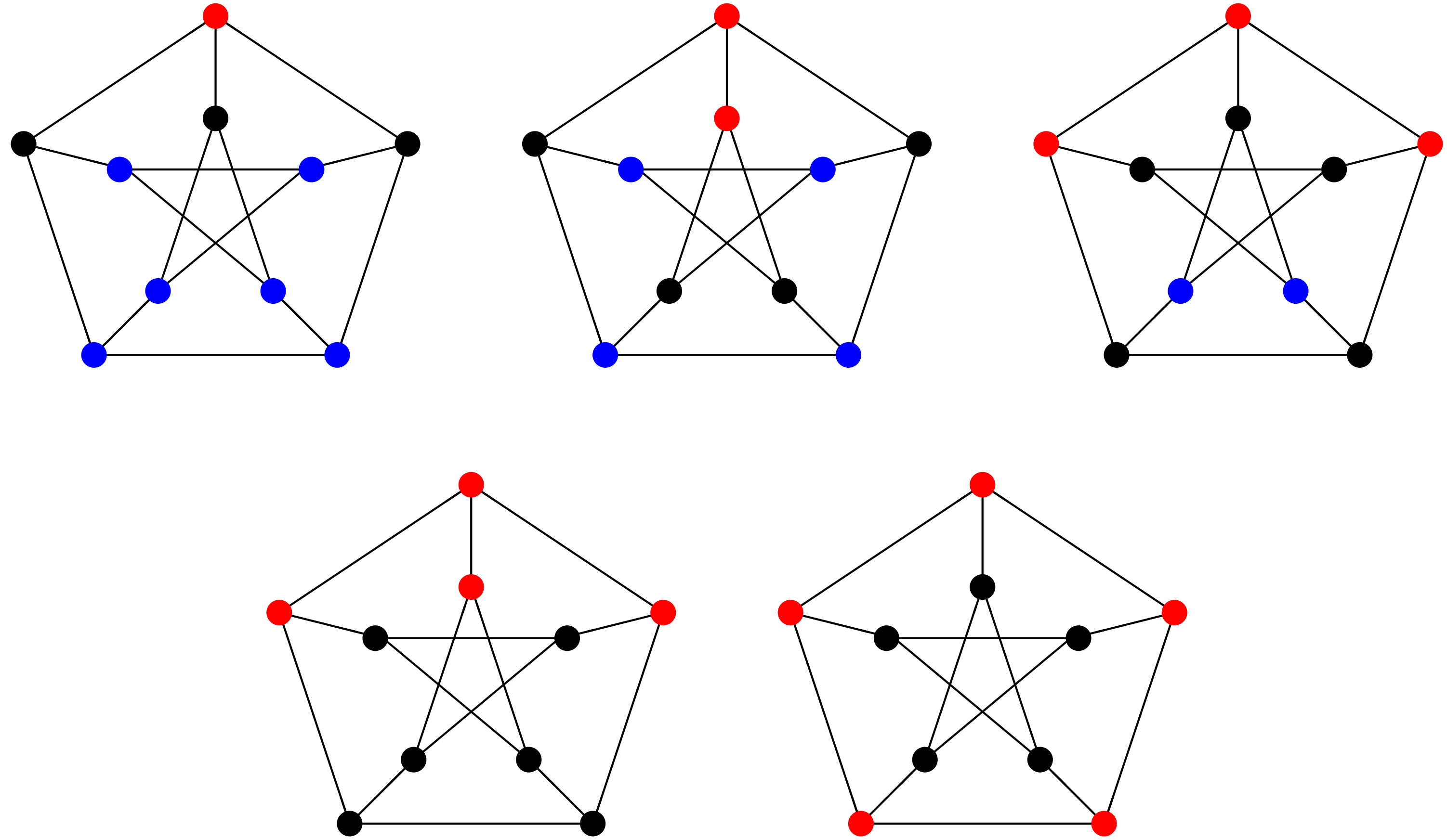
As in the case of sextics, a bouquet is determined by the "germs", *i.e.*, a collection $\mathcal{S}$ of 3-element subsets $s \subset \mathfrak{S} := \{1, \ldots, 6\}$. Any collection is realized by a **sub**graph of Fn X. There are 14 full bouquets, with the counts

$$1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 7, 8, 20,$$

and we can proceed with the proof as in the case of quartics: all configurations with > 30 lines are known [D., 2019].

→ **Degrees** 8 **and** 10

Degree 10 is trickier:

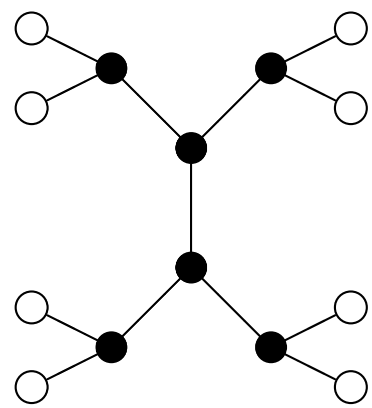


The situation is more involved: "germs" no longer suffice.
We need other means (discussed below).

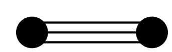
All degrees

To go further, need a systematic classification of $\hbar$-fragments. Use the taxonomy of hyperbolic graphs suggested in [D., 2019] (according to the minimal affine Dynkin sub-diagram):

- $\tilde{\mathbf{A}}_2$-, $\tilde{\mathbf{A}}_3$-, $\tilde{\mathbf{A}}_4$-, or $\tilde{\mathbf{A}}_5$-graphs; **one** section at each edge (starting from $\tilde{\mathbf{A}}_6$, cannot make cubic without a $\tilde{\mathbf{D}}_5$), or
- $\tilde{\mathbf{D}}_5$-graphs, with **all eight** simple sections.

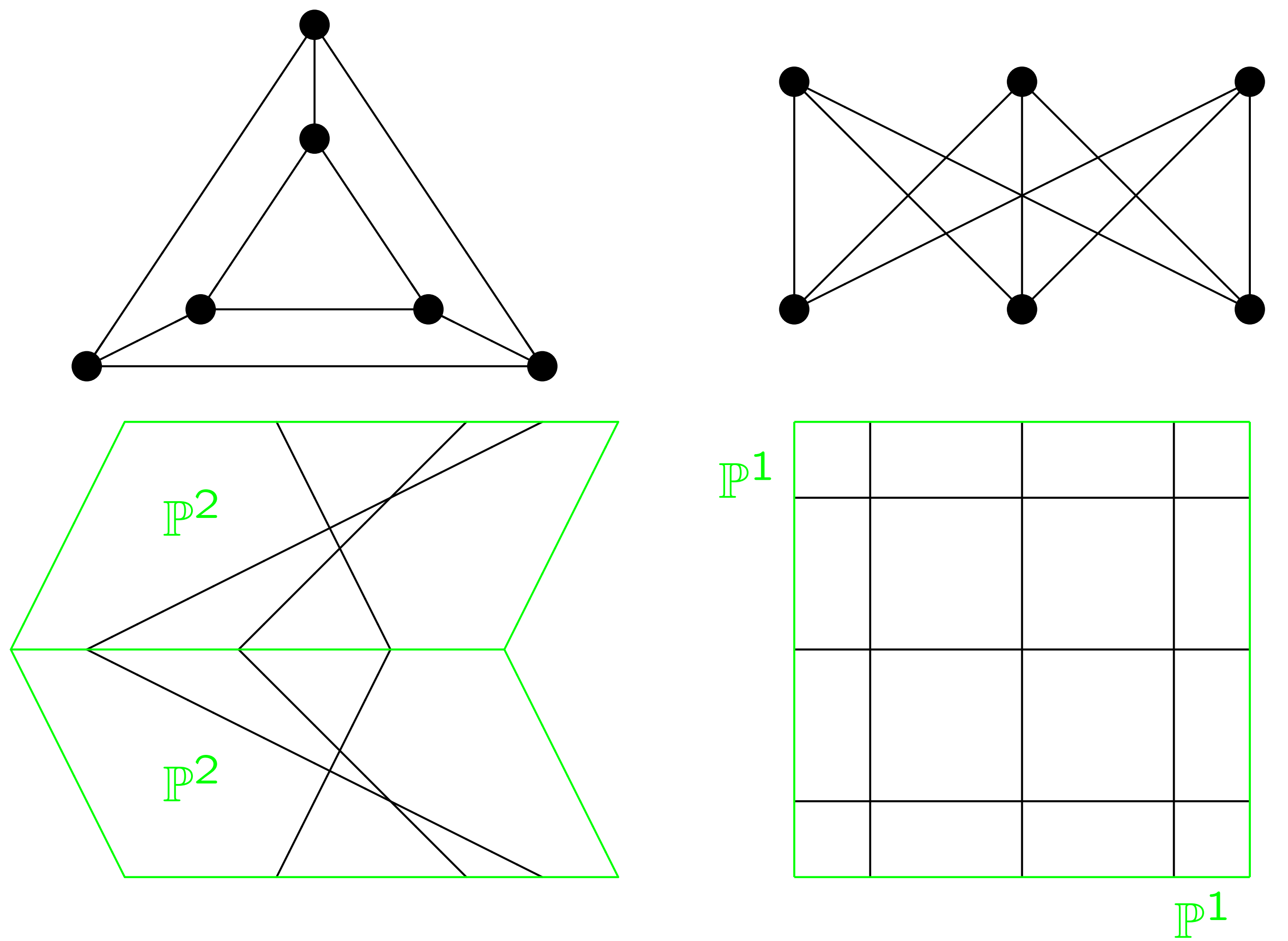


Use common sense (elliptic pencils) first, then the "magic."

Altogether, 48 simple graphs found, plus  for $\hbar^2 = 2$.

→ All degrees

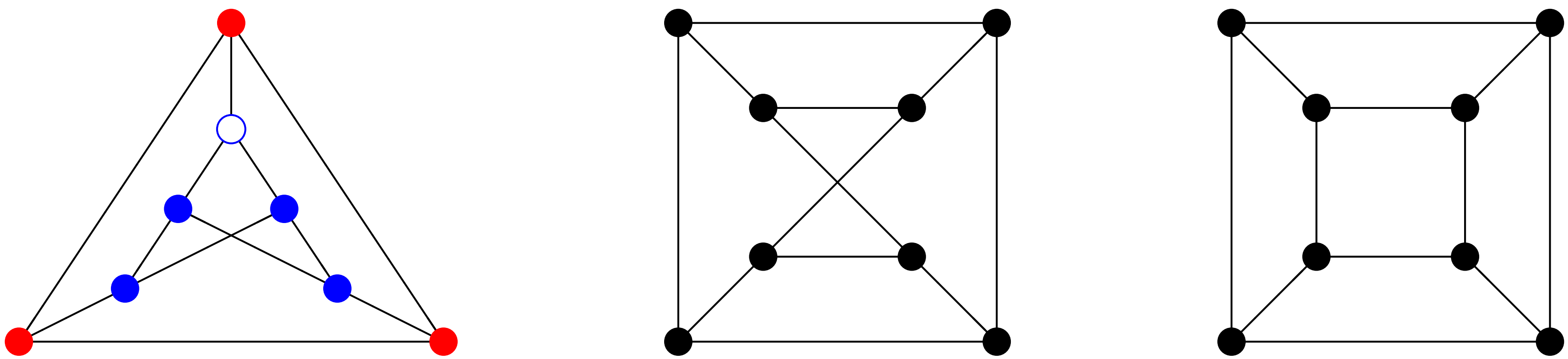
<u>*Sextics*: *two graphs* (max = 36 + 40 = **76**)</u>



$$\underbrace{0..6, 8, 9, 10, 12, 15, 16, 18, 20, 25, \underline{36} \qquad 0..24, 26, 27, 29, 30, 36, 40}_{0..34,\ 36,\ \underline{48},\ 49,\ 76}$$

→ All degrees

Octics: *three graphs* (max $= \mathbf{72}$ *or* $0 + 80 = \mathbf{80}$)



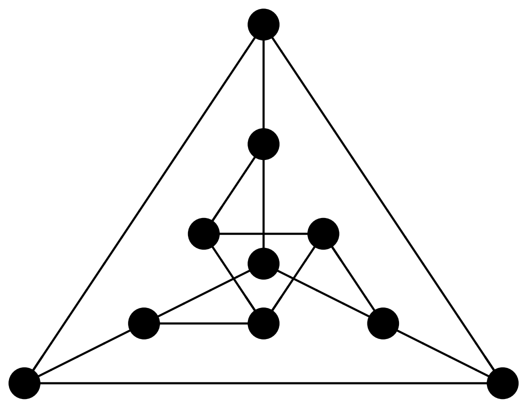
special	triquadric	triquadric
0..10, 12, 14, 16, 18, 20, 36, 42, 72	0..16, 18..21, 24, 32, <u>48</u>	0..13, <u>16</u>, 20, 21, 32, 80

(last two combined:) 0..18, 20, 21, 23, 24, 26, 36, 56, <u>64</u>, 80

Remark 9 *The last two: base locus of a net of quadrics in* $\mathbb{P}^4$. *Probably, Wagner means some sort of degeneration, too.* (*Fewer squares* ⇒ *fewer pairs of* $\mathbb{P}^3$.) *Still to be understood.*

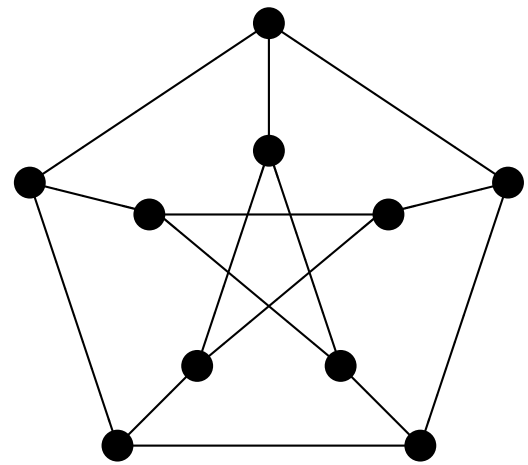
$\rightarrow$ All degrees

Degree $\hbar^2 = 10$: *six graphs* (max $= \mathbf{16}$)



$0, 1, 3, 6, 15$

$\cdots$



$0, \ldots, 8, 11, 14, 16$

Problem 10 *What does this all mean geometrically? Same about higher degrees.*

Remark 11 *Starting from* $\hbar^2 = 10$, *distinct* $\hbar$*-fragments do not "mix" very well.*

$\rightarrow$ **All degrees**

The principal result

Theorem 12 *The numbers of $\hbar$-fragments and maximal total counts are as follows*:

$\hbar^2 = 2d$	2	4	6	8	10	12	14	16	18	20	22	24	28
graphs	1	1	2	3	6	9	8	8	5	3	1	1	1
max #	72	72	76	80	16	90	12	24	3	4	1	1	1

Proof: for $\hbar^2 \geqslant 6$, it is easier to list all *$\hbar$-configurations*, *i.e.*, unions of $\hbar$-fragments; then, everything can be studied.
We add a *whole $\hbar$-fragment at a time*, increasing the rank fast.
There are restrictions on $\Gamma \cup \Sigma$ similar to $\Sigma_1 \cup \Sigma_2$ above.

For $\hbar^2 \geqslant 14$, easier to list all configurations (a line at a time).
Still there are restrictions $\Rightarrow$ converges fast. $\square$

Thank you!

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Marginal notes

1

/ 0: Wren, 1669; Shukhov, 1880's.

Cayley, 1849; Salmon, 1862

2

/ 0: *Fano graph*

3

/ 0: not graph!