Augmentable ordered abelian groups and definable henselian valuations

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Motivations

Question (L. S. Krapp, S. Kuhlmann, M. Link '23)

Describe the following classes:

$$D_{(\emptyset)-field} = \{k \text{ field} \mid \text{if } (K, v) \text{ is a henselian valued field with } Kv = k, \text{ then } v \text{ is } (\emptyset)\text{-definable in } \mathfrak{L}_{rings}\}.$$

$$D_{(\emptyset)-oag} = \{\Gamma \text{ oag} \mid \text{if } (K, v) \text{ is a henselian valued field with } vK = \Gamma, \text{ then } v \text{ is } (\emptyset)\text{-definable in } \mathfrak{L}_{rings}\}.$$

Definition

A field k is t-henselian if $k \equiv k'$ for some k' which admits a non-trivial henselian valuation.

Motivations

Theorem (A. Fehm, F. Jahnke '15)

Let (K, v) be a henselian valued field with non separably closed residue field. If Kv is not t-henselian then v is $\emptyset$-definable in $\mathfrak{L}_{ring}$.

Theorem (F. Jahnke, J. Koenigsmann '17)

Let K be a field such that $K \neq K^{sep}$. Then K admits a non-trivial definable henselian valuation unless $Kv_K \neq Kv_K^{sep}$ and Kv_K is t-henselian and $v_K K$ is divisible.

As a Corollary they prove a characterization theorem for fields with residue characteristic 0 admitting a non-trivial definable henselian valuation, generalized by M. Ketelsen, S. Ramello and P. Szewczyk (2024) for arbitrary residue field.

Motivations

Theorem (S. Anscombe, A. Fehm '17)

Let F be a field. TFAE:

- *there is a $\exists$-$\mathfrak{L}_{ring}$-formula that defines O_v [respectively $\mathfrak{m}_v$] in K for some non trivially valued henselian field (K, v) with residue field F;*
- *there is a $\exists$-$\mathfrak{L}_{ring}$-formula that defines O_v [respectively $\mathfrak{m}_v$] in K for every henselian valued field (K, v) with residue field elementarily equivalent to F;*
- *there is no elementary extension $F \preceq F^*$ with a non-trivial valuation v on F^* such that F^*v embeds in F^* [resp. with a non-trivial henselian valuation on a subfield E of F^* such that $Ev \cong F^*$].*

Remark. t-henselian $\Longrightarrow$ embedded residue + large.

Example

Let K be a field such that $char(K) = 0$ and $K \equiv K((\mathbb{Z}))$
(e.g. $K = \mathbb{Q}((\bigoplus_\omega \mathbb{Z}))$, by $\bigoplus_\omega \mathbb{Z} \equiv \mathbb{Z} \oplus \bigoplus_\omega \mathbb{Z}$)

Consider $\Gamma = \bigoplus_{\omega^*} \mathbb{Z}$ and $(K((\Gamma)), v_\Gamma)$.

Then v_Γ is **not** $\emptyset$-definable in $\mathfrak{L}_{ring}$:

$$L = K \underbrace{((\mathbb{Z}))\overbrace{((\Gamma))}^{u}}_{w}$$

is such that

$$(L, u, \Gamma, K((\mathbb{Z}))) \equiv (L, w, \Gamma \oplus \mathbb{Z}, K) \equiv (K((\Gamma)), v_\Gamma, \Gamma, K)$$

(by $K \equiv K((\mathbb{Z}))$, $\Gamma \equiv \Gamma \oplus \mathbb{Z}$ and Ax-Kochen/Ershov)

Augmentable ordered abelian groups

Definition

An oag G is Augmentable by Infinites if there is a non-trivial oag Δ such that $G \preceq \Delta \oplus G$.

By the results of Schmitt ('82) and Cluckers-Halupczok (2011), many properties of ordered abelian groups can be reduces to properties of their *spines*.

Definition

The spines $\mathcal{A}(G)$ is a collection of uniformily definable families of convex subgroups of G, ordered by inclusion and equipped with some unary predicates (colours) for first-order properties (e.g. discr,...).

Augmentable ordered abelian groups

Fact (Cluckers-Halupczok/Schmitt)

Let G, G' be ordered abelian groups. Then

$$G \equiv G' \iff \mathcal{A}(G) \equiv \mathcal{A}(G').$$

Moreover, if $G \subseteq G'$, then

$$G \preceq G' \iff \mathcal{A}(G) \preceq \mathcal{A}(G').$$

Lemma (F. Delon, F. Lucas '89)

Let $G \preceq G'$ be oags and $H = \langle G \rangle_{G'}$ the convex hull of G in G'. Then

$$G \preceq H \preceq G'.$$

Augmentable ordered abelian groups

Theorem (Boisonneau, DM, Jahnke, Touchard)

All non-trivial ordered abelian groups are Augmentable by Infinites.

Proof.

Let G be a non-trivial oag and consider an elementary extension G' that realizes a type at $+\infty$. Take $H = \langle G \rangle_{G'}$, then $G \preceq H \preceq G'$ and $G'/H \neq \{0\}$. Take $(G', H) \preceq (G^*, H^*)$ saturated such that the exact sequence

$$0 \longrightarrow H^* \longrightarrow G^* \longrightarrow G^*/H^* \longrightarrow 0$$

splits. Since $H \preceq H^*$, then $G \preceq H^*$. So we have

$$G^*/H^* \oplus G \preceq G^*/H^* \oplus H^* \cong G^*.$$

Since $G \subseteq G^*/H^* \oplus G$ and $G \preceq G^*$, then $G \preceq G^*/H^* \oplus G$. □

Definability by properties of the residue field

Lemma (Boissonneau, DM, Jahnke, Touchard)

Let k be a field with $\mathrm{char}(k) = 0$. If k is t-henselian, then $k \preceq k((\Delta))$ for some non-trivial oag Δ.

Sketch.

We may assume k saturated. By Prestel-Ziegler k admits a non-trivial equicharacteristic 0 valuation v. We may assume (k, v) saturated, so that $(k, v) \preceq (kv((vk)), u)$ by Ax-Kochen/Ershov. Set $\Gamma := vk$.
By the theorem $\Gamma \preceq \Delta \oplus \Gamma$ for some non-trivial oag Δ. Then

$$k \preceq kv((\Gamma)) \preceq kv((\Delta \oplus \Gamma))$$

and

$$k((\Delta)) \preceq kv((\Gamma))((\Delta)) \cong kv((\Delta \oplus \Gamma)).$$

It follows that $k \preceq k((\Delta))$. □

Definability by properties of the residue field

Theorem (Boissonneau, DM, Jahnke, Touchard)

Let k be a field with $\mathrm{char}(k) = 0$. TFAE:

1. *k is not t-henselian;*
2. *every henselian valuation v with residue field $k_v = k$ is definable in $\mathfrak{L}_{ring}$;*
3. *every henselian valuation v with residue field $k_v = k$ is $\emptyset$-definable in $\mathfrak{L}_{ring}$;*
4. *there is a $\emptyset$-$\mathfrak{L}_{ring}$-formula uniformly defining every henselian valuation v with residue field $k_v \equiv k$.*

Definability by properties of the residue field

Sketch.

2) $\Rightarrow$ 1). Assume k t-henselian. Then $k \preceq k((\Delta))$ for some non-trivial oag Δ.
By Tarski chain lemma, we have

$$k \preceq k((\Delta)) \preceq k((\Delta_{-1} \oplus \Delta_0 \oplus \Delta_{+1})) \preceq \ldots \preceq k((\bigoplus_{\mathbb{Z}} \Delta)).$$

Let $\Gamma := \bigoplus_{\mathbb{Z}} \Delta$, and consider $K = (k((\Gamma)), v)$. Note that $\Gamma \preceq \Gamma \oplus \Gamma'$, where Γ' is a copy of Γ (holds by playing Ehrenfeucht–Fraïssé games). Then

$$K^* = k \underbrace{((\Gamma')) \overbrace{((\Gamma))}^{w}}_{u}$$

is such that $K \preceq (K^*, w)$ and $K \preceq (K^*, u)$, i.e. v has residue field k and it is not definable in $\mathfrak{L}_{ring}$. □

Definability by properties of the value group

Definition

An oag G is augmentable by infinitesimals if there is a non-trivial oag Δ such that $G \preceq G \oplus \Delta$.

Proposition

Let G be an oag. If G is augmentable by infinitesimals, then there is a henselian valuation with value group G which is not definable in $\mathfrak{L}_{rings}$.

Question1. Does the converse hold?
Question2. Characterize oags augmentable by infinitesimals.

Theorem (Boissonneau, DM, Jahnke, Touchard)

Let (K, v) be an equicharacteristic 0 henselian valued field with value group G and residue field k. Then v is not definable in $\mathfrak{L}_{ring}$ if and only if there is a non-trivial oag Δ such that $G \preceq G \oplus \Delta$ and $k \preceq k((\Delta))$.

About characteristic p

Question3. What about residue fields of characteristic p?
Problem. Let k be a perfect field. Then $k \preceq k((\Delta))$ implies $k((\Delta))$ tame, since Δ is p-divisible (F.V. Kuhlmann '16). But there are perfect fields with no elementary extension which admit a tame valuation (e.g. $\left(\mathbb{F}_p((t))^{\mathcal{U}}\right)^{perf}$). We have

$k \preceq k((\Gamma))$ for some non-trivial ordered abelian group Γ

$\Downarrow$

there is a henselian valued field (K, v) with $Kv = k$ such that v is not definable

$\Downarrow$

there is a henselian valued field (K, v) with $Kv = k$ such that v is not $\emptyset$-definable

$\Downarrow$

henselian valuations with residue field elementarily equivalent to k are not uniformly $\emptyset$-definable in $\mathcal{L}_{ring}$

$\Downarrow$

k is t-henselian.