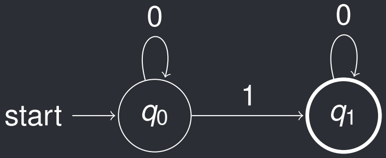
A Hierarchy of Expressive Power for Büchi Automata



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Automata Terminology

For an alphabet Σ, let Σ^* denote all strings Σ generates.

Call any subset $L \subseteq \Sigma^*$ a language. Say that an automaton A recognizes L if for all $w \in \Sigma^*$, running A on input w ends in an accept state iff $w \in L$. If L is recognized by some automaton, call it regular.

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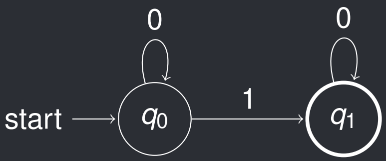

Example: The language this automaton recognizes is

$$L = 0^*10^* = \{0^n10^m : n, m \in \mathbb{N}\}.$$

Büchi automata

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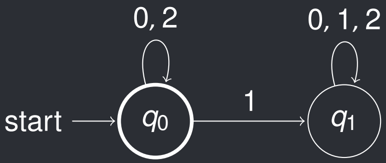

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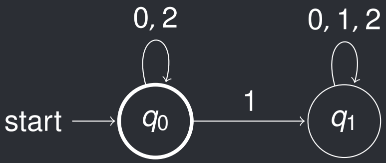

View the input strings for this automaton as the ternary representations for points in $[0,1]$, i.e. if $x = d_1\frac{1}{3} + d_2\frac{1}{9} + \ldots$ (with digits $d_1, d_2, \ldots \in \{0,1,2\}$) then "$d_1 d_2 \ldots$" is the input.

Say that $X \subseteq [0,1]$ is *k*-regular if there is a BA that accepts an input iff the input is a base-k expansion of some $x \in X$.

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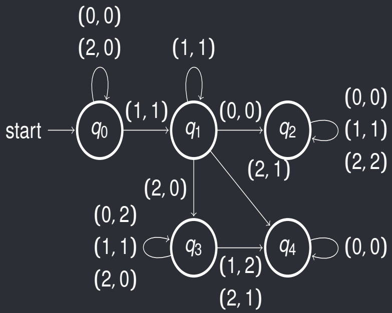

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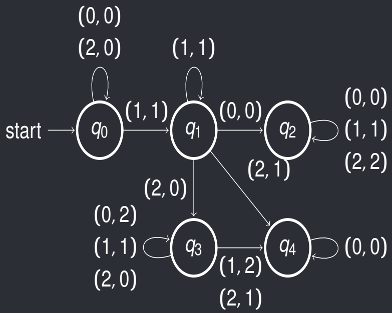
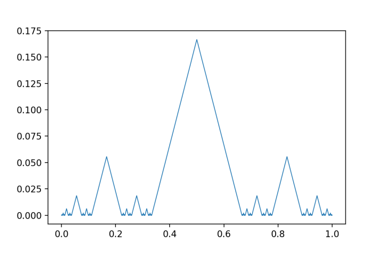
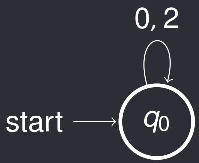


$X \subseteq [0,1]^2$ is k-regular if there is a Büchi automaton that accepts the sequence $(d_1^x, d_1^y), (d_2^x, d_2^y), \ldots \iff \exists (x, y) \in X$ such that $0.d_1^x d_2^x \ldots$ is the base-k representation of x and $0.d_1^y d_2^y \ldots$ is the base-k representation for y.

Fractals & Automata





This automaton recognizes $\mathscr{C}$, the Cantor set:

Above is the set recognized by the automaton on the previous slide.

Connection to first order logic

Definition

Let $V_k(x, u, d)$ be a relation on $\mathbb{R}^3$ that holds precisely if $u = k^{-n}$ for some $n \in \mathbb{N}_{>0}$ and the n^{th} digit of a base-k representation of x is d.

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Theorem (Boigelot, Rassart & Wolper, ’98)

A subset $X \subseteq [0, 1]^n$ is k-regular iff X is $\emptyset$-definable in $(\mathbb{R}, <, 0, +, V_k)$.

Corollary

The theory of $(\mathbb{R}, <, 0, +, V_k)$ is decidable.

Entropy

Definition

Given $A \subseteq \Sigma^*$, define the entropy of A as follows:

$$h(A) = \limsup_{n\to\infty} \frac{\log_k |A \restriction_n|}{n}$$

where $A \restriction_n$ is the set of length-n elements of A.

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Examples:

- If $k = 3$, then $h(\{0, 2\}^*) = \log_3(2)$.
- If $k = 2$, then $h(\{(0^*10^*1)^*0^*\}) = 1$.

Hierarchies

For $k \in \mathbb{N}_{>1}$, let $V_k(x, u, d)$ be a function on $\mathbb{R}^3$ that tells us the n^{th} digit in the base-k representation of $x \in [0, 1]$ is d.

	Decidable?	Geometry?	Every k-automatic set?
$(\mathbb{R}, <, +)$	Yes	o-minimal	No
$(\mathbb{R}, <, +, k^{\mathbb{N}})$	Yes	d-minimal	No
$(\mathbb{R}, <, +, \mathbb{Q}_{(k)})$	Yes	o-min open core	No
???	Yes	tame open core?	No
$(\mathbb{R}, <, +, V_k)$	Yes	No	Yes

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On $\mathbb{N}$, let $V_k(x)$ be the largest power of k that divides x.

	Decidable?	Entropies?	Every k-automatic set?
$(\mathbb{N}, +)$	Yes	{0,1}	No
$(\mathbb{N}, +, k^{\mathbb{N}})$	Yes	{0,1}	No
$(\mathbb{N}, +, V_k)$	Yes	Dense in $(0, 1)$	Yes

Closed sets & Automata

Say that a trim Büchi automaton $\mathscr{A}$ is closed if every state in $\mathscr{A}$ is accepting.

Let $\overline{\mathscr{A}}$ be the automaton resulting from making every state of $\mathscr{A}$ accepting.

Fact

If $\mathscr{A}$ recognizes $X \subseteq \mathbb{R}^d$, then $\overline{\mathscr{A}}$ recognizes $\overline{X}$, the topological closure of X.

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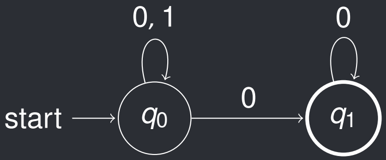

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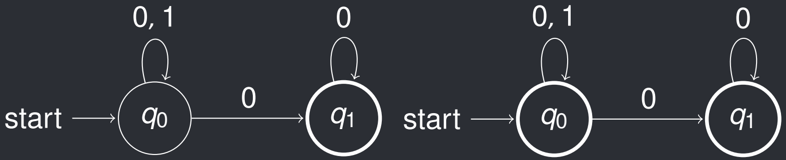

k-sparse languages

We say a set X is k-sparse if X is k-regular and the # of length n prefixes of elements of X grows at most polynomially in n.

Equivalently, $X \subseteq \mathbb{R}^m$ is a finite union of sets whose base-k representations are of the form

$$u_1 v_1^* \dots \bullet\, u_i v_i^* \dots u_d v_d^\omega$$

where $u_i, v_i \in (\Sigma^m)^*$ for each $i \le d$, and $\bullet$ is the k-adics point.

Examples:

$$k^{-\mathbb{N}} = .0^* 10^\omega$$

$$\frac{1}{k-1} - k^{-\mathbb{N}} = .1^* 01^\omega$$

Non-examples:

The Cantor set $\mathscr{C} = 0\bullet\{0,2\}^\omega$

$$\mathbb{D} := \left\{\frac{m}{2^n} : m < 2^n\right\} = \{0,1\}^* 0^\omega$$

Definability and $k^{-\mathbb{N}}$

Theorem (van den Dries, '85)

The structure $(\mathbb{R}, <, +, 2^{\mathbb{Z}})$ *is decidable and every unary set is a finite union of intervals and discrete sets (d-minimal).*

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Theorem (Bell-B. G.)

If $X \subseteq [0, 1]^d$ is k-sparse and infinite, there exists $m \in \mathbb{N}$ such that the structures $(\mathbb{R}, <, +, 0, k^{-m\mathbb{N}})$ and $(\mathbb{R}, <, +, 0, 1, X)$ define the same sets.

Model Theory & Definability

Suppose $X \subseteq [0,1]$ is k-regular. Let $L_X^{\mathrm{pre}} \subseteq \Sigma^*$ denote the set of all prefixes of base-k representations of elements of X. Call $\emptyset \neq C \subseteq \mathbb{R}$ a Cantor set if it is compact, has no isolated points, and no interior.

Theorem (Bell-B.G.)

If X is a closed k-regular subset of $[0,1]$ such that $0 < h(L_X^{\mathrm{pre}}) < 1$, then $(\mathbb{R}, <, +, 0, 1, X)$ defines a Cantor set.

Corollary (Bell-B. G.)

If $X \subseteq [0,1]^d$ is k-regular, closed, and interior-less, either $\mathrm{Def}(\mathbb{R}, <, +, 0, 1, X) = \mathrm{Def}(\mathbb{R}, <, +, 0, k^{-m\mathbb{N}})$ for some $m \in \mathbb{N}$, or $(\mathbb{R}, <, +, 0, 1, X)$ defines a Cantor set.

Open core

Given any topological structure $\mathscr{R}$, let $\mathscr{R}^\circ$ denote the structure

$$(R, (U)_{U \subseteq \mathscr{O}(\mathscr{R})})$$

where $\mathscr{O}(\mathscr{R})$ is all open definable subsets of R^n for each $n \in \mathbb{N}$, i.e. the predicates U range over the open sets of all arities definable in $\mathscr{R}$. Call this structure $\mathscr{R}^\circ$ the open core of $\mathscr{R}$.

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Examples:

- $\mathscr{R}$ o-minimal $\implies$ $\mathrm{Def}(\mathscr{R}^o) = \mathrm{Def}(\mathscr{R})$.
- The open core of $(\mathbb{R}, <, +, \mathbb{Q}_{(k)})$ defines the same sets as $(\mathbb{R}, <, +)$.
- The structure $(\mathbb{R}, <, +, (k^{m\mathbb{Z}})_{m \in \mathbb{N}})$ defines the same sets as its open core.

Hierarchy Revisited

Theorem (Balderrama-B. G.-Farris-Hieronymi-Manthe, 2025+)

For $X \subseteq \mathbb{R}$ a k-regular set, $(\mathbb{R}, <, +, X)^\circ$ defines no dense/codense set iff all definable sets in $(\mathbb{R}, <, +, X)^\circ$ are also definable in $(\mathbb{R}, <, +, (k^{m\mathbb{Z}})_{m\in\mathbb{N}})$.

Corollary

The open core of $(\mathbb{R}, <, +, \mathbb{Q}_{(k)}, k^{m\mathbb{Z}})_{m\in\mathbb{N}})$ defines the same sets as $(\mathbb{R}, <, +, (k^{m\mathbb{Z}})_{m\in\mathbb{N}})$.

Hierarchy Revisited

With this theorem, can characterize what the possibilities are for the "Geometry" of $(\mathbb{R}, <, +, X)$ for k-regular $X \subseteq \mathbb{R}$:

Reduct	Decidable?	Geometry?	All k-regular sets?
$(\mathbb{R}, <, +)$	Yes	o-minimal	No
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$(\mathbb{R}, <, +, \mathbb{Q}_{(k)})$	Yes	o-min open core	No
e.g., $(\mathbb{R}, <, +, \mathbb{Q}_{(k)}, k^{\mathbb{Z}})$	Yes	d-min open core	No
$(\mathbb{R}, <, +, V_k)$	Yes	No	Yes