Projective curves and weak second order logic

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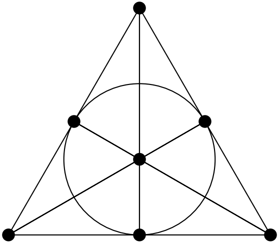
Projective plane

K is a field.

$\mathbb{P}^2(K) = K^3 \setminus \{0\}/\sim$ is the projective plane over K.



$$= \ \mathbb{P}^2(\mathbb{F}_2)$$

Instead of points and lines we want to consider points and irreducible curves of any degree.

A poset of points and curves

Let $\mathrm{Var}(K) = (\mathrm{Var}(K), \subset)$ be the poset of all non-empty Zariski closed irreducible proper subsets of $\mathbb{P}^2(K)$ ordered by inclusion.

$\mathrm{Var}(K)$ is a poset of height 2: the minimal elements are the points of $\mathbb{P}^2(K)$, the maximal elements are the irreducible projective curves $C \subset \mathbb{P}^2(K)$. The order is just the inclusion of a point in a curve.

$\mathrm{Var}_n(K) \subset \mathrm{Var}(K)$ is the substructure obtained by considering the points of $\mathbb{P}^2(K)$ and the curves of degree $\leq n$.

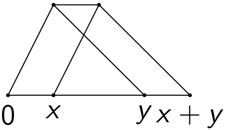
So $\mathrm{Var}_1(K)$ consists just of points and lines.

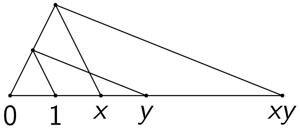
The poset $\mathrm{Var}_1(K)$ interprets the field K

Fix a line $L_\infty \in \mathrm{Var}_1(K)$ to play the role of the line at infinity.

By removing L_∞ and its points, we obtain a substructure $\mathrm{affVar}_1(K) \subset \mathrm{Var}_1(K)$ where we have a notion of parallelism (affine points and lines).

Now fix $L \in \mathrm{affVar}_1(K)$ and points $0, 1$ on L. We can define field operations on the points of L by using patterns of parallel lines.





The field thus obtained on the points of L is isomorphic to K.

Tressl's questions

Does Var($\mathbb{C}$) interpret $\mathbb{C} = (\mathbb{C}, +, \cdot)$?

Does Var($\mathbb{C}$) interpret $(\mathbb{Z}, +, \cdot)$?

We prove that Var($\mathbb{C}$) interprets $(\mathbb{C}, +, \cdot, \mathbb{Z})$, but it is stronger than that.

We also show that Var($\mathbb{C}$) $\not\equiv$ Var($\overline{\mathbb{Q}}$) despite the fact that $\mathbb{C} \equiv \overline{\mathbb{Q}}$.

Moreover Var($\mathbb{C}$) is recursively axiomatized modulo the theory of $\mathbb{Z}$.

A characterization of curves of degree ≤ 2

From now on, unless otherwise stated, assume that K is algebraically closed of characteristic zero.

For a curve $C \in \mathrm{Var}(K)$, the following are equivalent:

1. C has degree $d \leq 2$;
2. $\forall p \in C$ there is a curve $D \in \mathrm{Var}(K)$ such that $C \cap D = \{p\}$.

This is stated in [Davis-Maroscia 1984] (in the affine case), but depends on other references that we were not able to find.

For our proof we follow a suggestion of Rita Pardini: if C is smooth and 2 holds, then $\mathrm{Jac}(C)$ is torsion and the genus g is zero. In the smooth case $g = (d-1)(d-2)/2$.

We then use generalized Jacobians to prove that a curve C satisfying 2 is smooth.

$\mathsf{Var}(K)$ interprets $K = (K, +, \cdot)$

By the above, $\mathsf{Var}_2(K)$ is definable in $\mathsf{Var}(K)$.

Then $\mathsf{Var}_1(K)$ is also definable: C is a line if and only if it intersects every curve of degree ≤ 2 in at most two points.

Since $\mathsf{Var}_1(K)$ interprets $K = (K, +, \cdot)$, so does $\mathsf{Var}(K)$.

$\mathrm{Var}(K)$ interprets $(K, \mathrm{Fin}(K))$

Let $\mathrm{Fin}(X)$ be the finite power set of X.

Consider the two sorted structure $(K, \mathrm{Fin}(K))$ in a language with the ring operations on K and the membership relation between the two sorts.

We interpret $(K, \mathrm{Fin}(K))$ in $\mathrm{Var}(K)$ as follows.

Fix an affine line $L \in \mathrm{affVar}_1(K) \subset \mathrm{Var}_1(K)$.

A finite subset $S \in \mathrm{Fin}(L)$ can be coded by a curve $C \in \mathrm{Var}(K)$ that intersects L in the given subset S.

On the other hand L has definable ring operations making it isomorphic to $(K, +, \cdot)$, so $(K, \mathrm{Fin}(K))$ is interpretable in $\mathrm{Var}(K)$.

$\mathbb{Z} \subset K$ is definable in $(K, \mathrm{Fin}(K))$

Let K be an arbitrary field and consider the set $N_0 \subset K$ of all $x \in K$ such that there is $F \in \mathrm{Fin}(K)$ containing $0, x$ and such that, for all $z \in F$

- $z \neq 0 \implies z - 1 \in F$.
- $z \neq x \implies z + 1 \in F$.

Then:

K has characteristic zero $\iff -1 \notin N_0$

In this case, $N_0 = \mathbb{N}$ and $N_0 \cup -N_0 = \mathbb{Z}$.

In particular the characteristic zero property is definable in $(K, \mathrm{Fin}(K))$. Granted this property, $\mathbb{Z} \subset K$ is definable.

$(\mathbb{C}, \mathrm{Fin}(\mathbb{C}))$ is stronger than $(\mathbb{C}, \mathbb{Z})$

Return to our standand assumption when K is algebrically closed field of characteristic zero.

We have seen:

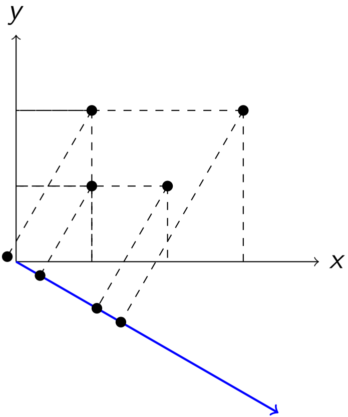
- $(K, \mathrm{Fin}(K))$ is interpretable in $\mathrm{Var}(K)$,
- $\mathbb{Z}$ is definable in $(K, \mathrm{Fin}(K))$.

When $K = \mathbb{C}$, the structure $(K, \mathrm{Fin}(K))$ is stronger than $(K, \mathbb{Z})$:

If $x \in \mathbb{C}$ is transcendental, the subgroup $x^{\mathbb{Z}} \subset K$ is not definable in $(\mathbb{C}, \mathbb{Z})$ [Toffalori-Vozoris 2010], but it is definable in $(\mathbb{C}, \mathrm{Fin}(\mathbb{C}))$.

Coding finite sets of pairs

If K is an infinite field we can interpret $(K, \mathrm{Fin}(K^2))$ in $(K, \mathrm{Fin}(K))$ following a suggestion of Mamino.



We code $S \in \mathrm{Fin}(K^2)$ by $(a, b, A, B, C) \in K^2 \times \mathrm{Fin}(K)^3$ where:

- $S \subset A \times B$;
- $(x, y) \in A \times B \mapsto ax + by \in K$ is injective;
- $C = \{ax + by \mid (x, y) \in S\}$.

Viceversa, given (a, b, A, B, C), we can define $S \in \mathrm{Fin}(K^2)$ by $(x, y) \in S :\iff (x, y) \in A \times B \wedge ax + by \in C$.

This gives the desired intepretation.

Coding finite sequences

There are infinite structures A such that $(A, \mathrm{Fin}(A))$ has a decidable theory.

On the other hand $(A, \mathrm{Fin}(A^2))$ always interprets $(\mathbb{Z}, +, \cdot)$.

If K is a field of characteristic zero, then $(K, \mathrm{Fin}(K))$ defines $\mathbb{Z} \subset K$ and interpretes $(K, \mathrm{Fin}(K^2))$, so we can code finite sequences of elements of K because such a sequence is an element of $\mathrm{Fin}(\mathbb{Z} \times K) \subset \mathrm{Fin}(K^2)$.

In general, in $(K, \mathrm{Fin}(K))$ we can code hereditarily finite sets and sequences of any sort.

For instance we can iterate Fin() and interpret $(K, \mathrm{Fin}(\mathrm{Fin}(\mathbb{Z} \times \mathrm{Fin}(K^5 \times \mathrm{Fin}(K^3)))$.

Var(K) is bi-interpretable with (K, Fin(K))

Return to the standard assumptions: K algebrically closed of characteristic zero.

Since in $(K, \mathrm{Fin}(K))$ we can code finite sequence, we can do recursive definitions and define the map $(x, n) \in K \times \mathbb{Z} \mapsto x^n$.

Similarly, we can then code polynomials of arbitrary degree and define a function taking the code of an homogeneous polynomial $p \in K[x, y, z]$ (of any degree) and a point $[a : b : c] \in \mathbb{P}^2(K)$ to the value $p(a, b, c) \in K$.

We can define a projective curve as an equivalence class of homogeneous polynomials with the same zeros, thus obtaining an interpretation of $\mathrm{Var}(K)$ in $(K, \mathrm{Fin}(K))$. One can check that this is in fact a bi-interpretation.

$\mathrm{Var}(\mathbb{C}) \not\equiv \mathrm{Var}(\overline{\mathbb{Q}})$

In $(K, \mathrm{Fin}(K))$ we can say that K has infinite transcendence degree:

$\forall S \in \mathrm{Fin}(K)\ \exists x \in K$ such that x is not algebraic over S
(we can quantify over codes of polynomials).

Via the bi-interpretation we then obtain $\mathrm{Var}(\overline{\mathbb{Q}}) \not\equiv \mathrm{Var}(\mathbb{C})$.

Towards an axiomatization

Our final goal is to show that $(\mathbb{C}, \mathrm{Fin}(\mathbb{C}))$ (hence also $\mathrm{Var}(\mathbb{C})$) is recursively axiomatizable modulo the theory of $\mathbb{Z}$.

Since the notion of finite set is not first order definable, a structure elementary equivalent to $(K, \mathrm{Fin}(K), \in)$ need not be isomorphic to one of the form $(L, \mathrm{Fin}(L), \in)$.

It turns out that a necessary (but not sufficient) condition for $(L, Y, \in^*) \equiv (K, \mathrm{Fin}(K), \in)$ is that Y is a definable finite power set of L in the sense of the next slide.

Some (incomplete) axioms for finite sets

Given a structure $\mathcal{A}$, let $\in^*$ be a definable binary relation in $\mathcal{A}$. Put

$$X = \mathrm{dom}(\in^*), \qquad \mathsf{F}_{\in^*}(X) = \mathrm{Img}(\in^*).$$

We say that $\mathsf{F}_{\in^*}(X)$ is a definable finite power set of X if:

- (*extensionality*) $\forall\ B, C \in \mathsf{F}_{\in^*}(X)$,

$$B = C \iff \forall x \in X.\ x \in^* B \iff x \in^* C;$$

- (*empty set*) there is $\emptyset^* \in \mathsf{F}_{\in^*}(X)$ such that

$$\forall x \in X.\ x \notin^* \emptyset^*;$$

- (*union with singletons*) Given $A \in \mathsf{F}_{\in^*}(X)$ and $c \in X$, there is $B \in \mathsf{F}_{\in^*}(X)$ (written as $B = A \cup^* \{c\}^*$) such that $\forall x \in X$:

$$x \in^* B \iff x \in^* A \vee x = c.$$

- (*set induction scheme*) for every definable $\mathcal{U} \subseteq \mathsf{F}_{\in^*}(X)$:

$$\begin{aligned} &\emptyset^* \in \mathcal{U} \\ &\forall x \in X.\ \forall A \in \mathsf{F}_{\in^*}(X).\ (A \in \mathcal{U} \implies A \cup^* \{x\}^* \in \mathcal{U}) \\ \implies &\mathcal{U} = \mathsf{F}_{\in^*}(X) \end{aligned}$$

Non-standard notions

Consider a structure of the form $(K, \mathsf{F}_{\in^*}(K))$ where $\mathsf{F}_{\in^*}(K)$ is a definable finite power set of K.

All the notions defined in $(K, \mathrm{Fin}(K))$ admit a non-standard counterpart in $(K, \mathsf{F}_{\in^*}(X))$, obtained by replacing $\in$ with $\in^*$ and every quantification over $\mathrm{Fin}(K)$ with a quantification over $\mathsf{F}_{\in^*}(X)$.

The axioms we have given for the notion of definable finite power set are strong enough to show that these non-standard notions behave well.

For instance if $(K, \mathsf{F}_{\in^*}(K))$ has characteristic zero in the non-standard sense, the definition of $\mathbb{Z}$ in $(K, \mathrm{Fin}(K))$ becomes, when interpreted in $(K, \mathsf{F}_{\in^*}(X))$, a definition of a subring $\mathsf{Z} \subset K$ whose positive part is a model of PA (although in general $\mathsf{Z} \neq \mathbb{Z}$).

Similarly, we can define non-standard polynomials and evaluate them.

The complete theory

A structure $(K, Y, \in^*)$ is elementary equivalent to $(\mathbb{C}, \mathrm{Fin}(\mathbb{C}), \in)$ if and only if:

1. K is a field.
2. Y is a definable finite power set of K.
3. K has characteristic zero in the non-standard sense.
4. K is algebraically closed in the non-standard sense.
5. K has infinite transcendence degree in the non-standard sense.
6. The non-standard integers $\mathrm{Z} \subset K$ of K are elementary equivalent to the standard integers $\mathbb{Z}$.

Warning: there are structures of the form $(\mathbb{C}, \mathrm{F}_{\in^*}(\mathbb{C}))$ where $\mathbb{C}$ does not have characteristic zero in the non-standard sense.

Ingredients of the proof

The proof of completeness is by back and forth between models with the same non-standard integers.

We need to show that the type of an element $x \in K$ is determined by the set of non-standard polynomials over the non-standard rationals which have x as a root.

This is proved via a version of Hilbert's basis theorem for non-standard polynomials: every definable polynomial ideal over a definable subring of K has a non-standard finite basis.

We also need to determine the type of an element of $\mathsf{F}_{\in^*}(K)$.

To realize such a type in the other model, we need a secondary internal back and forth of non-standard length!

Stable embeddedness

We can show that every subset of $\mathbb{Z}^n$ parametrically definable in $(\mathbb{C}, \mathsf{Fin}(\mathbb{C}))$ is already definable in $(\mathbb{Z}, +, \cdot)$

By contrast, in the structure $(\mathbb{R}, \mathsf{Fin}(\mathbb{R}))$ one can define (with parameters) every subset of $\mathbb{Z}$.

A quote from Hartshorne

If you like the purely synthesic approach to geometry, you may enjoy von Staudt's treatment of the quadratic surfaces in projective three-space, extending even to a discussion of the twisted cubic curve. But I would be the first to admit that in higher dimensions, and for varieties of higher degree the synthetic methods become overly cumbersome and that modern algebraic methods are more convenient.

Foundations of projective geometry (2009 edition)