Valuations on groups, ordered groups, and exponential groups

by Vincent Bagayoko (Université Paris Cité)

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Are there an $\mathcal{N} = (N, \dots) \models T$ with $\mathcal{M} \subseteq \mathcal{N}$ and an $a \in N$ such that

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The term $t(y) = g_1\, y^{\alpha_1} \cdots g_n\, y^{\alpha_n}$ is said **regular** if $\alpha(t) := \alpha_1 + \cdots + \alpha_n \neq 0$.

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Expansions of groups:

- Ordered groups: same as for orderable.
- "Exponential groups": no systematic study of this problem.

Definition: A **group** is a group.

Notations: $(\mathcal{G}, \cdot, 1)$; $\mathcal{G}^{\neq} := \mathcal{G} \setminus \{1\}$; $[f, g] := f^{-1} g^{-1} f g$; $\mathcal{C}(f) := \{g \in \mathcal{G} : [f, g] = 1\}$.

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An **ordered group** *is a group* $(\mathcal{G},\cdot,1,<)$ *together with a* *total* *ordering* $<$ *on* $\mathcal{G}$ *with*

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for all $f,g,h\in\mathcal{G}$*. We write* $\mathcal{G}^{>} = \{g\in\mathcal{G} : g>1\}$.

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Definition: exponential group (Miasnikov&Remeslennikov, 1994)

Let A *be a (unital, associative) ring. An* **A-group** *is a group* $(\mathcal{G},\cdot,1)$ *together with a function* $A\times\mathcal{G}\longrightarrow\mathcal{G}\,;\,(a,g)\mapsto g^a$ *such that for all* $g,h\in\mathcal{G}$ *and* $a,b\in A$, *we have:*

$g^0=1$, $g^1=g$, $1^a=1$	$g^{a+b}=g^a g^b$, $(g^a)^b=g^{ab}$
$(hgh^{-1})^a = hg^ah^{-1}$	$[g,h]=1\Rightarrow(gh)^a=g^ah^a$.

Examples: groups are $\mathbb{Z}$-group in a unique way: $(n,g)\mapsto g^n$. Divisible groups with unique roots are $\mathbb{Q}$-groups in a unique way.

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- **valued** if for all $f, g, h \in \mathcal{G}$,

  V1. $f \neq 1 \rightarrow v(1) < v(f)$.

  V2. $v(fg) \leqslant \max(v(f), v(g))$.

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- **c-valued** if it is valued and for all $f, g \in \mathcal{G}$,

  V5. $(g\,f = f\,g \wedge f \neq 1 \wedge g \neq 1) \rightarrow v(g) = v(f)$.

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- **A-valued** if it is valued and for all $a \in A$ and $f, \varepsilon \in \mathcal{G}$,

$$f^a \neq 1 \Rightarrow v(f^a) = v(f) \qquad \text{and} \qquad v(\varepsilon) < v(f) \Rightarrow v((f \varepsilon)^a f^{-a}) \leqslant v(\varepsilon).$$

Let F be a field and let V be a vector space over F. The group $\mathrm{Aff}_F(V)$ of affine bijections of V is the group under composition of maps $\lambda x + u : v \mapsto \lambda v + u$ for $\lambda \in F^\times$ and $u \in V$.

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Valuation $v : \mathrm{Aff}_F(V) \longrightarrow (\{0, 1, 2\}, <)$ with

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\begin{aligned}
v(x) &= 0 \\
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This is a c-valuation on $\mathrm{Aff}_F(V)$.

Free groups

Proposition

Let $\mathcal{F}$ be a non-Abelian free group. Let $(\mathcal{F}_n)_{n<\omega}$ and $(\mathcal{F}^{(n)})_{n<\omega}$ denote the lower central and derived series of $\mathcal{F}$ respectively. Then $\mathcal{F}$ is c-valued and $\mathbb{Z}$-valued for the maps

$$\ell : g \mapsto \sup\{n \in \mathbb{N} : g \in \mathcal{F}_n\} \qquad \textit{and} \qquad \mathrm{d} : g \mapsto \sup\{n \in \mathbb{N} : g \in \mathcal{F}^{(n)}\}.$$

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Proof (for ℓ). We take the reverse ordering on $\Gamma = \omega + 1$. $\mathbf{V1}$ holds by definition. $\mathbf{V2}$ and $\mathbf{V3}$ hold since each $\mathcal{F}_n$ is a subgroup. $\mathbf{V4}$ holds because each $\mathcal{F}_n$ is normal. $\mathbf{V6}$ holds because each quotient $\mathcal{F}_n/\mathcal{F}_{n+1}$ is Abelian. □

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Now consider $\mathbf{V5}$. Let $f, g \in \mathcal{F}^{\neq}$ with $g \in \mathcal{C}(f)$. We have $\mathcal{C}(f) = s^{\mathbb{Z}}$ for a certain $s \in \mathcal{C}(f)$. So it suffices to show that $v(s^m) = v(s)$ for all $s \in \mathcal{F}^{\neq}$ and $m \in \mathbb{Z} \setminus \{0\}$. Note that $\mathcal{F}_{v(s)}$ is a free group. Assume for contradiction that $s^m \in \mathcal{F}_{v(s)+1}$. Then $(s\,\mathcal{F}_{v(s)+1})^m = 1$ in the free Abelian group $\mathcal{F}_{v(s)} / \mathcal{F}_{v(s)+1}$. It follows that $s \in \mathcal{F}_{v(s)+1}$: a contradiction. Thus $\mathbf{V5}$ holds and $\mathcal{F}$ is c-valued. This plus normality of each $\mathcal{F}_n$ in $\mathcal{F}$ implies it is $\mathbb{Z}$-valued. □

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By work of Baumslag, Jaikin-Zapirain and Miasnikov&Remeslennikov, this generalises to free A-groups for commutative domains A of characteristic 0.

Definition

A **growth order group** *is an ordered group* $(\mathcal{G},\cdot,1,<)$ *such that*

- *The map* $v:\mathcal{G}\longrightarrow 2^{\mathcal{G}}$ *given by* $v(f)=\mathrm{Convex\,Hull\,of}(\mathcal{C}(f))$ *is nondecreasing on* $\mathcal{G}^{>}$.
- *We have* $f\,g>g\,f$ *for all* $f,g\in\mathcal{G}^{>}$ *with* $v(f)>v(g)$.
- *For all* $\gamma\in v(\mathcal{G}^{\neq})$, *there is an* f *with* $v(f)=\gamma$ *such that* $\mathcal{C}(f)$ *is Abelian and that for all* $g\in\mathcal{G}$ *with* $v(g)=\gamma$, *there is an* $f_0\in\mathcal{C}(f)$ *with* $v(g\,f_0^{-1})<\gamma$.

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Theorem

If $\mathcal{R}$ *is a polynomially bounded o-minimal expansion of* $(\mathbb{R}, +, \cdot)$, *then the ordered group under composition of germs at* $+\infty$ *of* $\mathcal{R}$*-definable unary functions* f *with* $\lim f = +\infty$ *is a growth order group.*

This also works if for each definable map f with $\lim f = +\infty$ there are an $e \in \mathbb{Z}$, $n \in \mathbb{N}$ with

$$\exp^{\circ e} - 1 \leqslant \log^{\circ n} \circ f \circ \exp^{\circ n} \leqslant \exp^{\circ e} + 1.$$

Let $(\mathcal{G}, \cdot, 1, v)$ be c-valued, let $\gamma \in v(\mathcal{G}^{\neq})$. Write

$$\mathcal{G}_{\leqslant \gamma} = \{f \in \mathcal{G} : v(f) \leqslant \gamma\}, \qquad \mathcal{G}_{< \gamma} = \{f \in \mathcal{G} : v(f) < \gamma\} \lhd \mathcal{G}_{\leqslant \gamma}, \qquad \text{and} \qquad \mathcal{C}_{\gamma} = \mathcal{G}_{\leqslant \gamma} / \mathcal{G}_{< \gamma}.$$

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Then $\mathcal{C}_{\gamma}$ is Abelian. If $\mathcal{G}$ is an A-valued A-group, then $\mathcal{C}_{\rho}$ is an A-module. The union of quotient maps $\bigsqcup_{\gamma \in v(\mathcal{G}^{\neq})} \mathcal{G}_{\leqslant \gamma} \setminus \mathcal{G}_{<\gamma} \longrightarrow \bigsqcup_{\gamma \in v(\mathcal{G}^{\neq})} \mathcal{C}_{\gamma}$ is denoted res.

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$$\begin{array}{ccccccccc} 0 & \hookrightarrow & \mathcal{C}_{\gamma_1} \simeq \mathcal{G}_{<\gamma_2} & \hookrightarrow & \mathcal{G}_{\leqslant\gamma_2} & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{G}_{\leqslant\gamma_n} = \mathcal{G} \\ & & & & \downarrow & & & & \downarrow \\ & & & & \mathcal{C}_{\gamma_2} & & & & \mathcal{C}_{\gamma_n} \\ & & & & \downarrow & & & & \downarrow \\ & & & & 0 & & & & 0 \end{array}$$

Spherical completeness

Definition

A valued group is said **spherically complete** *if all decreasing families of valuative balls* $g\mathcal{G}_{<\gamma}$, $g \in \mathcal{G} \wedge \gamma \in v(\mathcal{G}^{\neq})$ *have non-empty intersection.*

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- *Valued groups* $(\mathcal{G}, v)$ *with finite value set* $v(\mathcal{G})$, *e.g. torsion-free nilpotent groups with lower central, or derived valuation.*
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I don't know if a valued group can always be embedded into a spherically complete one.

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Examples: free groups with the lower central valuation; groups of contracting derivations; the growth order group of *parabolic* germs definable in a polynomially bounded o-minimal expansion of $(\mathbb{R}, +, \cdot)$, i.e. germs of the form $\mathrm{id} + \delta$ where $\frac{\delta}{\mathrm{id}} \to 0$.

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In general, I expect most interesting spherically complete ordered valued groups can be obtained as direct limits of inverse limits of semidirect products of nearly Abelian ones.

Results on nearly Abelian exponential groups (I)

Fix a ring A of characteristic 0, and a nearly Abelian A-valued A-group $\mathcal{G}$. Suppose that $\mathcal{G}$ is A-torsion-free: $g^a = 1 \Rightarrow (a = 0$ or $g = 1)$.

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A unary term in the (one-sorted) language of A-groups "is" an element of the free product of A-groups $\mathcal{G} *^A A$. We have a unique A-group homomorphism $\alpha : \mathcal{G} *^A A \longrightarrow A$ with $\alpha \restriction \mathcal{G} = 0$ and $\alpha \restriction A = \mathrm{id}_A$. A term t is said *regular* if $\alpha(t) \neq 0$.

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Lemma: residue of a regular term

For any regular $t \in \mathcal{G} *^A A$ *with* $t(1) = 1$ *and all* $f \in \mathcal{G}$*, we have* $\mathrm{res}(t(f)) = \alpha(t)\,\mathrm{res}(f)$.

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Proof for $A = \mathbb{Z}$. Write $t = g_1\, y^{\alpha_1} \cdots g_n\, y^{\alpha_n}$, so $\alpha(t) = \alpha_1 + \cdots + \alpha_n$. Since $t(1) = 1$, we can write $t = t_1 \cdots t_n$ where $t_i := (g_1 \cdots g_i)\, y^{\alpha_i}\, (g_1 \cdots g_i)^{-1}$. Near Abelianness: $\forall f, g \in \mathcal{G}^{\neq}$, $v(f\, g\, f^{-1}\, g^{-1}) < v(g)$ so $\mathrm{res}(f\, g\, f^{-1}) = \mathrm{res}(g)$ for all $f, g \in \mathcal{G}$. Thus for all $f \in \mathcal{G}$, we have $\mathrm{res}(t_i(f)) = \alpha_i\, \mathrm{res}(f)$. Since $\alpha_1 + \cdots + \alpha_n \neq 0$, we have $\mathrm{res}(t(f)) = \alpha(t)\, \mathrm{res}(f)$. □

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Consequence: we can solve $t(y) = 1$ for regular terms t *via* iterated approximations ($\eta \in \mathbf{On}$):

- $t_\eta(1) = 1$, then the only solution is $y = 1$.
- if $t_\eta(1) \neq 1$, then repeat by considering the term $t_{\eta+1} = t_\eta(f_\eta\, y)$ where $\alpha(t)\,\mathrm{res}(f_\eta) + \mathrm{res}(t(1)) = 0$ in $\mathcal{C}_{v(t_\eta(1))}$.
- at limit stages λ, take a "pseudo-limit" of $(f_0\, f_1 \cdots f_\eta \cdots)_{\eta < \lambda}$.

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Corollary: extension of an old result of Smel'kin (1967)

If $\mathcal{H}$ is residually A-nilpotent A-torsion free, then any regular $t\in\mathcal{H}*^A A$ has a solution in the residually A-nilpotent A-torsion free completion $\tilde{\mathcal{H}}$ ($\tilde{\mathcal{H}}=\mathcal{H}$ if it is A-nilpotent A-torsion-free).

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What's next?

Spherical completion, divisible extensions of residues, singular equations (commutators, conjugacy, ...), the case of non-nearly Abelian c-valued groups.

Thanks!

Let F be a field, $\chi(F)=0$. Let G be an Abelian ordered group. We have a Hahn series field $F((G))$, with its valuation $v : F((G))^{\times} \longrightarrow G$. A linear map $\phi : F((G)) \longrightarrow F((G))$ is **contracting** if $v(\phi(s)) > v(s)$ for all $s \neq 0$.

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Let $\partial: F((G)) \longrightarrow F((G))$ be a *strongly linear* derivation with $\mathrm{Ker}(\partial) = F$. Then $F((G))$ is a Lie algebra for $[\![\cdot,\cdot]\!]: (f, g) \mapsto \partial(f)\, g - f\, \partial(g)$. Set

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$(\mathrm{Cont}(\partial), *, 0, v)$ *is an F-valued, c-valued group for the Baker-Campbell-Hausdorff product*

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Proposition

$\mathrm{Cont}(\partial)$ *is a growth order group with valuation $-v$ if and only if $(F((G)), v, \partial)$ is an H-field.*